System of Josephson junctions as a model of a spin glass

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It is shown that a system of superconducting wires, connected by Josephson junctions and placed in a magnetic field, undergoes a transition to a low-temperature state of the spin-glass type. The dependence of the phase-transition temperature T_c on the magnetic field H is found. In the vicinity of T_c superconducting fluctuations lead to a logarithmic growth of the effective conductivity of the system with decrease of the frequency ω and of the relative temperature $\tau = T/T_c - 1$; the effective inductance is proportional to $\min(1/\omega, 1/\tau^2)$. It is shown that the low-temperature state of the system depends on the path along which it was reached in the (T,H) plane, the history-dependent equation of state is derived, and the existence of a diamagnetic response to a change of the magnetic field is predicted. An experiment making it possible to resolve the question of the existence of a phase transition in a vector spin glass is proposed.

1. INTRODUCTION

The only analytically studied model of a spin glass is the Sherrington-Kirkpatrick model [1,2] in which the interaction between the spins does not depend on the distance between them. However, in practically all spin glasses the interaction between the spins decreases rapidly with increase of the distance between them, and therefore the results of this model do not permit a direct quantitative comparison with experiment. Below, we shall study a model of a physical system which, on the one hand, can be prepared experimentally, and, on the other hand, is very similar, in one of its limiting cases, to the Sherrington-Kirkpatrick model and so admits an analytical solution. This physical system is a system of specially arranged superconducting junctions placed in an external magnetic field. Systems of this kind are convenient for modeling various phase transitions. For example, in Ref. 3 the effect of hierarchical structure on a phase transition was studied experimentally, while in Refs. 4 and 5 the influence of incommensurability was studied. A system of superconducting balls connected by Josephson links forming a percolation network was studied in Ref. 6. The same system, placed in a strong magnetic field, was considered by John and Lubensky in Ref. 7, in which it was shown that this system is equivalent to a spin glass with two-component spins. However, as in all the models enumerated above, in the John-Lubensky model only nearest neighbors interact, while in the Sherrington-Kirkpatrick model all the spins interact with one another.

In the present paper we proposeand investigate a model (analogous to a spin glass in which all the spins interact with one another) of a system of superconducting junctions in a magnetic field. It will be shown that in this system a macroscopically coherent state with randomly frozen phases of the order parameter and with diamagnetic response is realized at low temperatures. Possible generalizations of the model and its application to the investigation of spin glasses are discussed.

2. DESCRIPTION OF THE MODEL

To realize a physical system in which every element interacts with every other element is rather complicated; however, a system in which all the elements are divided into two groups such that each element from one group interacts with all the elements of the other is considerably simpler to make. We shall consider a model system consisting of N vertical and N horizontal superconducting filaments, arranged in two parallel planes in such a way that each vertical filament is connected by Josephson junctions with all the horizontal filaments, and vice versa. The distance between neighboring parallel filaments is assumed to be random, and its average value equal to l. We shall study the behavior of this system in a magnetic field perpendicular to the planes in which the filaments lie. We shall assume that the energies of the Josephson links of the different junctions are equal, but most of the results will also remain valid in the case when J fluctuates not very strongly from junction to junction. The Josephson junctions between the filaments can be both metallic and dielectric; for us it is important only that the energy J of these junctions is small— much smaller than the transition temperature T_{c0} of an individual superconducting filament. In addition, we neglect the effect of the magnetic field created by the superconducting currents flowing through the Josephson junctions in comparison with the effect of the external magnetic field (this corresponds to the condition $JN \ll H\hbar lc/e$). The superconducting filaments can be realized in different ways, it being important only that their thickness be smaller than the penetration depth of the magnetic field, so that their presence does not change the magnitude of the external magnetic field. In the region of extremely weak magnetic fields, such that the flux of the magnetic field through the entire system is much smaller than a quantum of flux, the effect of the magnetic field on the phase transition can be neglected, and in this region the transition occurs at the temperature $T_c = JN$. We shall be interested in the region of stronger fields:

$$(Nl)^2 Hl \gg \hbar c/e, \tag{2.1}$$

in which the influence of frustration of the links between different elements becomes important; on the other hand, the fields should be sufficiently weak, i.e., smaller than the critical field of an individual filament. At temperatures not very close to the superconducting-transition temperature of an individual filament the fluctuations of the modulus and nonuniform fluctuations of the phase of the order parameter

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can be neglected, and each filament is described entirely by its phase $\varphi(\mathbf{r})$, which depends only on the coordinate along the filament. The current flowing along an individual filament is small, and so

$$\nabla\varphi(\mathbf{r})=2e\mathbf{A}/\hbar c,$$

where **A** is the vector potential, which we choose in the gauge $A_y = Hx$ (x and y are the coordinates along the horizontal and vertical filaments, respectively, and we take the coordinate origin to be at the center of the system). In this gauge the phase of the order parameter of the horizontal filaments does not depend on the coordinate along the filament and is equal to $\varphi_i(x) = \varphi_i$, while the phase of the order parameter of a vertical filament is $\varphi_j(y) = x_j y l_H^{-2} + \varphi_j$ (x_j is the coordinate of the jth vertical filament, and l_H is the magnetic length $l_H^2 = \hbar c/2eH$). The phase difference across the junction between the ith horizontal and the jth vertical filament is equal to $\varphi_i - \varphi_j - x_j y_i l_H^{-2}$, the energy of such a contact is equal to $J\cos(\varphi_i - \varphi_j - x_j y_i l_H^{-2})$, and that part of the energy of the entire system which depends on the phases of the individual filaments is equal to

$$\mathscr{H}=-J\sum_{i,j}\cos(\varphi_i-\varphi_j-x_jy_il_H^{-2})=-\mathrm{Re}\sum_{ij}S_i^*J_{ij}S_j, \qquad (2.2)$$

where

$$S_i=\exp(i\varphi_i),\ J_{ij}=J\exp(ix_iy_jl_H^{-2}).$$

The Hamiltonian (2.2) is analogous to the Hamiltonian for a spin glass with a two-component XY spin. As in a spin glass, the average value of J_{ij} is equal to zero (when the condition (2.1) is fulfilled). However, in contrast to the case of spin glasses, the quantities J_{ij} are not independent for different pairs (i,j), and, therefore, in addition to the usual quantity $K_1 = \overline{J_{ij}^2}$ we shall also need averages of products of an arbitrary even number of J_{ij}:

$$K_m=\overline{J_{i_1j_1}J_{j_1i_2}\ldots J_{j_mi_1}}. \qquad (2.3)$$

Here repeated indices do not imply summation, and all indices appear exactly twice; the bar denotes averaging over the coordinates x_i and y_j; $J_{ij} = J_{ji}^*$. Performing this averaging, we obtain

$$K_m=(2\pi l_H^2/L^2)^{m-1}J^{2m}, \qquad (2.4)$$

where $L = Nl$ is the length of each filament.

3. SPECTRUM OF THE MATRIX J_{ij}

Before turning to the study of the physical characteristics of the system with the Hamiltonian (2.2), it is useful to study the spectrum of the random matrix $\hat{J}_{ij}$, whose eigenfunctions $\psi_i^{(\lambda)}$ and eigenvalues E_λ are specified in the usual way:

$$\sum_j \hat{J}_{ij}\psi_j^{(\lambda)} = E_\lambda\psi_i^{(\lambda)},\quad \hat{\mathbf{J}}=\begin{pmatrix}0 & J_{ij}\\ J_{ji}^* & 0\end{pmatrix}.$$

We introduce the corresponding Green function:

$$g_{ij}(E)=[\delta_{ij}E-\hat{J}_{ij}]^{-1}. \qquad (3.1)$$

Then the density of eigenvalues E_λ of the matrix $\hat{\mathbf{J}}$ is

$$\rho(E) = -\frac{1}{\pi}\,\mathrm{Im}\,\overline{g_{ii}(E+i0)}. \qquad (3.2)$$

The calculation of the average one-point Green function $g(E) = \overline{g_{ii}(E)}$ can be represented conveniently in diagrammatic form (see Fig. 1). Diagrams with intersecting dashed lines make a contribution of order $1/N$ (this is connected with the interaction, adopted in the model, of all the spins with all the spins). The corresponding analytical expression has the form

$$g(E)=(1+g(E)\Sigma)E^{-1}, \qquad (3.3)$$

where

$$\Sigma=\sum_{m=1}^{\infty}(Ng(E))^{2m-1}K_m. \qquad (3.4)$$

Using (2.4) and calculating the sum (3.4), we obtain for $g(E)$ the cubic equation

$$(E+i0)g(E)=1+J^2N\left(1-\frac{2\pi l_H^2}{l^2}g^2(E)\right)^{-1}g^2(E). \qquad (3.5)$$

In strong magnetic fields, $l_H \to 0$ and Eq. (3.5) reduces to a quadratic equation (well known in the theory of completely random matrices[8,9]), with a semicircular solution for the density of states (as $H \to \infty$):

$$\rho(E) = \frac{1}{2\pi NJ^2}(4NJ^2-E^2)^{1/2}. \qquad (3.6)$$

In weak fields ($l_H^2 \gg l^2 N$) the density of states has a peak at $E = 0$ and two domes near $E^2 = 2\pi l_H^2 J^2/l^2$:

$$\rho(E)=\left(1-\frac{Nl^2}{2\pi l_H^2}\right)\delta(E)+z^{1/2}\theta(z), \qquad (3.7)$$

where

$$z=2J^2N-\left(|E|-\frac{(2\pi)^{1/2}l_HJ}{l}\right)^2.$$

With increase of the magnetic field the amplitude of the δ-function peak decreases by the same law as in formula (3.7), and at the field

$$H=H_0=\pi\hbar c/el^2N \qquad (3.8)$$

vanishes. With increase of the field the side domes broaden and, at the field H_0, coalesce into one, which for $H \gg H_0$ takes the form (3.6). At the field H_0 there is one quantum of flux per strip between two neighboring parallel filaments.

The boundaries E_0 of the domes at arbitrary fields are

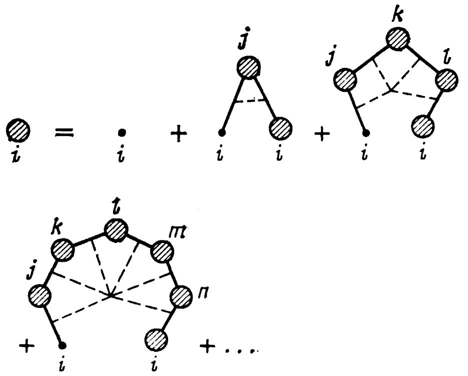

FIG. 1. Diagrammatic Dyson equation for the average one-point Green function (shown by a shaded circle). The heavy point corresponds to the bare Green function the heavy lines correspond to the quantities J_{ij}, and the dashed lines denote the averaging over the disorder.

determined by the appearance of an imaginary part in the solutions of Eq. (3.5). As $E\to E_0$ these solutions have a square-root singularity:

$$\partial g(E)/\partial E\propto(E-E_0)^{-1/2}.$$

Therefore, regarding Eq. (3.5) as determining the inverse function $E(g)$, we obtain $\partial E(g)/\partial g=0$, whence, using (3.5), we have the equation

$$NJ^2g^2(E_0)\frac{1+(2\pi l_H^2J^2/l^2)g^2(E_0)}{[1-(2\pi l_H^2J^2/l^2)g^2(E_0)]^2}=1. \tag{3.9}$$

The solution of this biquadratic equation has two (for $H>H_0$) or four real roots. Substituting them into (3.5), we find the boundaries of the spectrum. Below we shall be interested in the upper boundary E_0 of the spectrum. Solving (3.5) and (3.9) simultaneously, we obtain

$$E_0=\left[\frac{NJ^2}{2}\left(1+\frac{2H_0}{H}+\left(1+\frac{8H_0}{H}\right)^{1/2}\right)\right]^{1/2}$$
$$\times\left[1+\frac{H}{4H_0}\left(\left(1+\frac{8H_0}{H}\right)^{1/2}-1\right)\right]. \tag{3.10}$$

For $H\gg H_0$ the expression (3.10) reduces to $E_0=2JN^{1/2}$, corresponding to (3.6). Thus, the correlations between the matrix elements J_{ij} can be neglected for $H\gg H_0$. In the opposite limit $H\ll H_0$ (here we always have (2.1) in mind), we obtain from formula (3.10) or (3.7)

$$E_0=JN^{1/2}((H_0/H)^{1/2}+2^{1/2}). \tag{3.11}$$

When the field $H=H_0$, the boundary of the spectrum is equal to $E_0=3J(3N)^{1/2}/2$, and the density of states has a singularity at low energies:

$$\rho(E)\approx\frac{3^{1/2}}{2\pi}\left(\frac{J^2N}{E}\right)^{1/3}. \tag{3.12}$$

When the field $H\gtrsim H_0$, the density of states at $E=0$ is equal to

$$\rho(0)=\frac{1}{\pi(NJ^2(1-H_0/H))^{1/2}}. \tag{3.13}$$

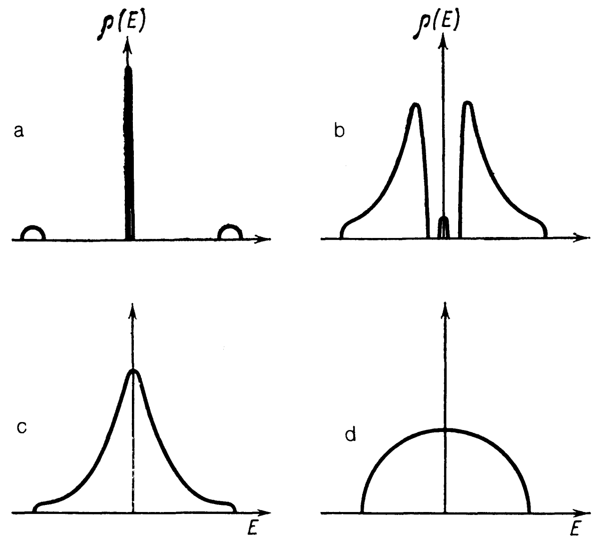

FIG. 2. Approximate form of the density of states $\rho(E)$ of the matrix $\hat{\mathbf{J}}$ for different values of H: a) $H\ll H_0$; b) $H_0-H\ll H_0$; c) $H-H_0\ll H_0$; d) $H\gg H_0$.

Qualitatively, the form of the spectrum for different fields is illustrated in Fig. 2.

4. THE TRANSITION TEMPERATURE

Even below the superconducting-transition temperature of an individual filament, thermal fluctuations of the phases can lead to disordering of the phases; the true phase transition occurs at a certain temperature $T_c<T_{c0}$. In a system of finite size a true phase transition is, of course, impossible, but the smearing of the phase transition is small in $1/N$; in the following we shall disregard effects of this kind. The transition temperature T_c is defined as the value of T at which a finite thermodynamic average value $m_i=\langle S_i\rangle$ appears in an infinitesimal fictitious external field h_i (this field is introduced by adding to the Hamiltonian (2.2) a term $-\mathrm{Re}\,h_i^*S_i$). To determine T_c we write for m_i, in the linear approximation, an equation analogous to the Thouless-Anderson-Palmer equation,[10] well known in the theory of spin glasses (see also Ref. 11 and the Appendix to Ref. 12):

$$m_i=\frac{1}{2T}\left(\sum_j J_{ij}m_j+h_i\right)-\alpha m_i. \tag{4.1}$$

The last term in (4.1) (the Bragg-Williams term) is the correlation correction to the usual mean-field equations; the necessity of taking this correction into account in glasses was first demonstrated in Ref. 10. For spin glasses with XY spins the quantity α was calculated in Refs. 11 and 12: $\alpha=(1/4T^2)\Sigma_j J_{ij}^2$. In our case this expression is valid only in the limit $H/H_0\to\infty$; the quantity α in arbitrary fields will be calculated somewhat later.

From (4.1) there follows an expression for the generalized susceptibility $\chi_{ij}=\partial m_i/\partial h_j$:

$$\chi_{ij}=[2T(1+\alpha)\hat{1}-\hat{J}]^{-1}\equiv g_{ij}(E=2T(1+\alpha)). \tag{4.2}$$

In the representation of the eigenfunctions $\psi_i^{(\lambda)}$ from (4.2) we obtain

$$\chi_\lambda=[2T(1+\alpha)-E_\lambda]^{-1}. \tag{4.3}$$

The phase transition occurs at that temperature at which the singularity in χ_λ first appears; i.e., the equation for T_c has the form

$$2T_c(1+\alpha(T_c))=E_0, \tag{4.4}$$

where the upper boundary E_0 of the spectrum is determined by the formulas (3.9) and (3.10). By virtue of (4.2) and (4.4) the quantity $\overline{\chi_{ii}(T_c)}$ coincides with the quantity $\overline{g_{ii}(E_0)}=g(E_0)$ introduced earlier (see (3.1)). On the other hand, the linear-response relation

$$2T\chi_{ij}=\langle\langle S_iS_j^*\rangle\rangle=\langle S_iS_j^*\rangle-\langle S_i\rangle\langle S_j^*\rangle$$

for "stiff spins" ($|S_i|=1$) in the high temperature phase leads to the equality

$$\chi_{ii}=1/2T. \tag{4.5}$$

Thus, we have

$$g(E_0)=\chi_{ii}(T_c)=1/2T_c.$$

Substituting this expression into (3.9), we obtain a quadratic equation for the quantity T_c^2:

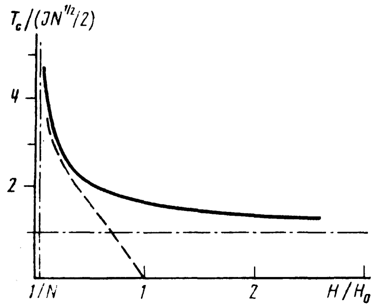

FIG. 3. Dependence of the transition temperature T_c on the magnetic field H (the solid curve). The dashed curve shows the conjectured line of a first-order phase transition at lower temperatures (see the Conclusion).

$$\frac{NJ^2}{4T_c^2}\frac{1+\varkappa}{(1-\varkappa)^2}=1, \qquad \varkappa=\frac{\pi l_H^2 J^2}{2T_c^2 l^2}, \tag{4.6}$$

the solution of which has the form

$$T_c=\frac{JN^{1/2}}{2^{3/2}}\left[1+\frac{2H_0}{H}+\left(1+\frac{8H_0}{H}\right)^{1/2}\right]^{1/2}. \tag{4.7}$$

A graph of the dependence (4.7) is presented in Fig. 3. We note that the approach to the limiting dependences

$$T_c=\frac{JN^{1/2}}{2} \tag{4.8}$$

for $H \gg H_0$ and

$$T_c=\frac{JN^{1/2}}{2}\left(\frac{H_0}{H}\right)^{1/2} \tag{4.9}$$

for $lH_0/L \ll H \ll H_0$ is extremely slow. Formula (4.8) coincides with the known formula for an XY spin glass, and corresponds to the absence of correlations between the quantities J_{ij} in a strong field.

To calculate the transition temperature we did not need the explicit form of the Bragg-Williams term in Eq. (4.1). However, this term can be of independent interest. It is equal to the sum of the diagrams of the one-loop approximation (analogous to the diagrams depicted in Fig. 1, in which the shaded circles correspond to the one-point susceptibilty, equal to $1/2T$ for the "stiff" model). Summing the diagrams by means of the formulas (2.4), in analogy with the result (3.5) we obtain

$$\alpha=\frac{NJ^2}{4T^2}\left(1-\frac{\pi l_H^2 J^2}{2l^2T^2}\right)^{-1}. \tag{4.10}$$

Substituting this value of α into formulas (4.2) and (3.5) for the one-point susceptibility, we obtain for the latter the value $1/2T$, as we should.

5. DYNAMICS OF THE MODEL ABOVE THE TRANSITION POINT

1. The properties of spin glasses manifest themselves most clearly in their low-frequency dynamics. Above the transition temperature the relaxation times are microscopic, but increase as the transition point is approached. Below the transition point certain times become macroscopically large. The physical quantity most convenient for measurement in the system under consideration is, as in ordinary spin glasses, the response of the system to an external magnetic field. However, in contrast to ordinary spin glasses, the magnetic field in this system does not act on the individual spins but changes the magnitude of the interaction between them.

We shall obtain first of all the equations of motion for the phases of the superconducting filaments. The total current I_{ij} flowing through the Josephson junction between a vertical and a horizontal superconducting filament (with phases φ_i and φ_j, respectively) is made up of three parts: a superconducting current, equal to

$$(2eJ/\hbar)\sin[\varphi_i-\varphi_j-(2eH(t)/\hbar c)x_iy_j],$$

a normal current, equal to

$$(\hbar/2eR)(\dot\varphi_i-\dot\varphi_j)-(\dot H/cR)x_iy_j$$

(where R is the resistance of the contact), and a Nyquist noise current ζ_{ij}. The quantity ζ_{ij} is a random Gaussian variable:

$$I_{ij}=\frac{\hbar}{2eR}\frac{d\varphi_{ij}}{dt}+\frac{2eJ}{\hbar}\sin\varphi_{ij}+\zeta_{ij}, \tag{5.1}$$

$$\langle\zeta_{ij}(t)\zeta_{kl}(t')\rangle=\delta_{ik}\delta_{jl}2T\delta(t-t'),$$

where

$$\varphi_{ij}=\varphi_i-\varphi_j-\frac{2eH(t)}{\hbar c}x_iy_j. \tag{5.2}$$

For the low-frequency dynamics under consideration, that part of the total current which is associated with the capacitance of the contacts is small, and so its contribution to the total current can be neglected. The equation of motion follows from the current-conservation law

$$\sum_j I_{ij}=0, \qquad \sum_i I_{ij}=0. \tag{5.3}$$

The part due to the normal current in Eqs. (5.3) is equal to

$$\left(N\dot\varphi_i-\sum_j\dot\varphi_j\right)\hbar/2eR.$$

The phases of different filaments are weakly correlated with each other; therefore

$$\left|\sum_j\dot\varphi_j\right|\sim\sqrt{N}|\dot\varphi_i|\ll N|\dot\varphi_i|$$

and the term $\Sigma_j\dot\varphi_j$ can be neglected. As a result, we obtain the final form of the dynamical equations:

$$\begin{aligned}&\frac{\hbar N}{2eR}\frac{d\varphi_i}{dt}+\sum_j\left(\frac{2eJ}{\hbar}\sin\varphi_{ij}+\zeta_{ij}\right)=0,\\&-\frac{\hbar N}{2eR}\frac{d\varphi_i}{dt}+\sum_i\left(\frac{2eJ}{\hbar}\sin\varphi_{ij}+\zeta_{ij}\right)=0.\end{aligned} \tag{5.4}$$

It will be convenient below to measure the time in dimensionless units:

$$dt/d\tilde t=t_r=\hbar^2N/4e^2RT. \tag{5.5}$$

(The temperature T of the thermostat can, generally speaking, be a function of time.) In the variables Eqs. (5.4) have the form

$$\begin{aligned}&\dot\varphi_i+\sum_j\left(\frac{J}{T}\sin\varphi_{ij}+\tilde\zeta_{ij}\right)=0,\\&-\dot\varphi_j+\sum_i\left(\frac{J}{T}\sin\varphi_{ij}+\tilde\zeta_{ij}\right)=0,\\&\langle\tilde\zeta_{ij}(t)\tilde\zeta_{kl}(t')\rangle=\frac{2}{N}\delta(t-t')\delta_{ik}\delta_{jl}.\end{aligned} \tag{5.6}$$

Here and below, the time t denotes the dimensionless variable $\tilde{t}$.

To investigate the system (5.4) it is convenient to write its formal solution in the form of a functional integral.[13] The average value of the current I_{ij} (5.1) flowing through the junction (i,j) is given in this formalism by the expression

$$\langle I_{ij}\rangle=\int \mathcal{D}\psi\mathcal{D}\varphi I_{ij}\exp\{A[\psi,\varphi]\}, \tag{5.7}$$

where, in the new variables,

$$I_{ij}=\frac{2e}{\hbar}\left(\frac{T}{N}\dot{\varphi}_{ij}+J\sin\varphi_{ij}\right). \tag{5.8}$$

The action A has the form $A=A_0+A_1$, where

$$A_0=\int dt\left[\sum_i(-\psi_i^2+i\psi_i\dot{\varphi}_i)+\sum_j(-\psi_j^2+i\psi_j\dot{\varphi}_j)\right],$$

$$A_1=i\int dt\left[\frac{J}{T}\sum_{ij}(\psi_i-\psi_j)\sin\varphi_{ij}\right]. \tag{5.9}$$

In A_0 we have neglected the term $\Sigma_{ij}\psi_i\psi_j/N$, which is small for large N.

In the action A we have not written out the term that arises from the Jacobian but does not contain the auxiliary field ψ. As usual, the role of these terms in the perturbation-theory series reduces to the cancellation of terms containing response functions G at equal times.[13]

The action A_1 can be written conveniently in the form

$$A_1=-\frac{1}{2T}\sum_{ij}\int dt(\psi_i-\psi_j)J_{ij}S_iS_j^*, \tag{5.10}$$

where J_{ij} and S_i are defined in the same way as in formula (2.2). The summation in (5.10) is such that the indices i, j run over the values for both the horizontal and the vertical filaments.

We first find the one-particle Green functions

$$G_{ij}(t,t')=-\langle S_i(t)\psi_j(t')S_j^*(t')\rangle,$$
$$D_{ij}(t,t')=\langle S_i^*(t)S_j(t')\rangle. \tag{5.11}$$

The Green function G describes the response of the system to a fictitious external field and is retarded: $G(t,t')=0$, if $t<t'$. In a constant field and above the transition temperature the system is in thermal equilibrium. Therefore, the functions G and D are connected by the fluctuation-dissipation theorem, which in the units (5.5) has the form

$$G_{ij}(t,t')=-\frac{\partial D_{ij}^*(t,t')}{\partial t}\theta(t-t'). \tag{5.12}$$

Expanding the expression (5.11) in a series in A_1, we obtain for the function G the Dyson equation

$$G_{ij}=G'\delta_{ij}+(J_{ik}/2T)G'G_{kj}, \tag{5.13}$$

where G' denotes the sum of the one-particle-irreducible Green functions, which can be written in the form of a series in powers of J^2:

$$G'=G_0+G_1'+\dots.$$

In zeroth order in J^2, with the action A_0, the Gaussian integral in (5.11) is easily calculated:

$$G_0(t)=e^{-t}\theta(t),\quad G_0(\omega)=1/(1-i\omega)\approx 1+i\omega. \tag{5.14}$$

In first order in J^2,

$$G_1'=\frac{J^2}{T}\sum_j\int[X(t,t_1,t_2,t')G_{jj}(t_1,t_2)$$
$$-2Y(t,t_1,t_2,t')D_{jj}(t_1,t_2)]dt_1\,dt_2, \tag{5.15}$$

where the irreducible correlators X and Y are equal to

$$X(t,t_1,t_2,t')=\langle S(t)\psi(t_1)(S^*(t_1)S(t_2)+\text{h.c.})\psi(t')S^*(t')\rangle_0$$
$$-G_0(t,t_1)G_0(t_2,t'), \tag{5.16}$$
$$Y(t,t_1,t_2,t')=\langle S(t)\psi(t_1)S(t_1)\psi(t_2)S^*(t_2)\psi(t')S^*(t')\rangle.$$

At large times t_1-t_2 the correlators X and Y decay like $\exp(-4(t_1-t_2))$, and therefore in the function G' the slow dynamics is unimportant. At low frequency

$$G_1'(\omega)=-\frac{J^2N}{4T^2}\left(1-\frac{i}{5}\omega\right)G_0^2(\omega). \tag{5.17}$$

In the static limit $\omega=0$, as follows from (5.12), the function G coincides with the single-time correlator D, which, in its turn, is proportional to the static susceptibility and can be found from the static formulas of the preceding section, whence

$$G'(\omega=0)=D'(t,t)=(1+\alpha)^{-1}, \tag{5.18}$$

where α is the TAP parameter,[10] equal to the expression (4.10), At low frequencies

$$(G'(\omega))^{-1}=1+\alpha-i\gamma\omega. \tag{5.19}$$

The difference of the parameter γ from unity takes into account the renormalization of the relaxation time on account of fast processes. As follows from formulas (5.14) and (5.17),

$$\gamma=1-\frac{1}{5}\frac{J^2N}{4T^2}+\dots\ . \tag{5.20}$$

In weak magnetic fields $H\ll H_0$, the parameter $J^2N/4T^2$ is small and γ is close to unity. In arbitrary fields and in the spin-glass limit $H\gg H_0$ the parameter $J^2N/4T^2$ near the transition point is of order unity and in the stiff model under consideration it is necessary to take into account all the terms in formula (5.20). We have not found a way of summing them, but there are no grounds to suppose that this sum has a singularity as $T\to T_c$. The small numerical factor 1/5 in formula (5.20) is connected with the rapid decay of the irreducible correlators X and Y, and it may be supposed that it will remain present in the next orders.

The physical relaxation is described by the full Green function G, which satisfies Eq. (5.13). This equation coincides with formula (3.1) after the replacement $G'\to E^{-1}$, $J_{ij}/2T\to J_{ij}$. The averaging reduces to summation of the diagrams of Fig. 1 and leads to Eq. (3.5), which now has the form

$$1+\alpha-i\gamma\omega=G^{-1}+\frac{J^2N}{4T^2}\frac{G}{1-(\pi l_H{}^2J^2/2T^2l^2)G^2}, \tag{5.21}$$

where $G=G(\omega)=\overline{G_{ii}(\omega)}\equiv 1+\delta(\omega)$.

At low frequencies the Green function differs little from unity ($\delta(\omega)\ll 1$) and satisfies a quadratic equation, which, near T_c, has the form

$$\delta^2+2\tau\delta+i\Gamma\omega=0, \tag{5.22}$$

where $\tau=(T-T_c)/T_c$ and $\omega,\tau\ll\Gamma$; for Γ we have

$$\Gamma=\gamma\frac{1-\varkappa^2}{1+3\varkappa^2}=\gamma\frac{[x+(x^2+8x)^{1/2}]^2}{[2+x+(x^2+8x)^{1/2}]^2+4x-4}. \quad (5.23)$$

Here, $\varkappa=\pi l_H^2 J^2/2T_c^2 l^2=(J^2N/4T_c^2)(H_0/H)$, and $x=H_0/H$; in deriving the second equality of (5.23) we took (4.7) into account. In the limiting cases for Γ we obtain

$$\Gamma=\gamma\approx 1\,(H\gg H_0),\qquad \Gamma=\left(\frac{H}{2H_0}\right)^{1/2}\,(H\ll H_0).$$

Solving (5.22) and going over to the time representation, we obtain

$$G(t)=\frac{1}{2}\left(\frac{\Gamma}{\pi}\right)^{1/2}\frac{1}{t^{3/2}}\exp\left(-\frac{t}{t_0}\right)\theta(t), \quad (5.24)$$

where $t_0=\Gamma\tau^{-2}$. We find the correlation function D ($\overline{D}_{ij}=\delta_{ij}D$) with the aid of (5.12):.

$$D(t)=\left(\frac{\Gamma}{\pi}\right)^{1/2}\left[\frac{1}{t^{1/2}}\exp\left(-\frac{t}{t_0}\right)+\left(\frac{\pi}{t_0}\right)^{1/2}\times\left(\operatorname{erf}\left(\left(\frac{t}{t_0}\right)^{1/2}\right)-1\right)\right], \quad (5.25)$$

where erf(x) is the error integral.

We emphasize that the formulas (5.24)–(5.25) are applicable only at sufficiently long times: $t\gg\Gamma^{-1}$.

We shall study the response of the system of junctions to a change of the field, which we represent as the sum of a constant part H_0 and a varying part $H_1(t)$:

$$H(t)=H_0+H_1(t).$$

The field change $H_1(t)$ will be assumed to be small in comparison with the constant part of the field.

A convenient macroscopic quantity related to the currents is the total magnetic moment created by them:

$$M=\frac{1}{c}\sum_{ij}I_{ij}x_iy_j=M_n+M_s,$$

where M_n and M_s are created by the normal current and the superconducting current, respectively. In the dimensionless variables that we are using the normal current is equal to

$$I_{ij}^{(n)}=-\dot{H}x_iy_j/c^2Rt_r,$$

and, therefore, it creates a magnetic moment

$$M_n=-\left(\frac{NL^2}{12}\right)^2\frac{\dot{H}(t)}{c^2Rt_r}. \quad (5.26)$$

To calculate the superconducting part of the magnetic moment we shall make use of the functional-integral formalism. We obtain

$$M_s=\frac{2e}{\hbar c}\sum_{i,j}x_iy_j\int\mathscr{D}\psi\mathscr{D}\varphi\,(J_{ij}S_iS_j^*-J_{ji}S_i^*S_j)\,\frac{1}{2i}\exp(A_0+A_1), \quad (5.27)$$

where the actions A_0 and A_1 are given by formulas (5.9) and (5.10), respectively, the sum over i in (5.27) runs over the vertical filaments, and the sum over j runs over the horizontal filaments. We shall average the expression (5.27) over the coordinates of the vertical and horizontal filaments. We obtain

$$M_s=\frac{2e}{i\hbar c}\sum_{ij}\int\mathscr{D}\varphi\mathscr{D}\psi\Big\{\frac{1}{2T}\int dt_1S_i(t)S_j^*(t)\times(\psi_j(t_1)-\psi_i(t_1))S_j(t_1)S_i^*(t_1)\overline{x_iy_jJ_{ij}(t)J_{ji}(t_1)}$$
$$+\frac{1}{(2T)^3}\sum_{kl}\int dt_1\,dt_2\,dt_3S_i(t)S_j^*(t)(\psi_j(t_1)-\psi_k(t_1))S_j(t_1)S_k^*(t_1)(\psi_k(t_2)-\psi_l(t_2))S_k(t_2)S_l^*(t_2)(\psi_l(t_3)-\psi_i(t_3))S_l(t_3)S_i^*(t_3)\times\overline{x_iy_jJ_{ij}(t)J_{jk}(t_1)J_{kl}(t_2)J_{li}(t_3)}+\ldots\Big\}\exp(A_0+A_{eff}), \quad (5.28)$$

where $A_{\rm eff}$ is the effective action obtained after averaging over the coordinates x_i, y_j. The condition $N\gg1$ enables us to make use of the mean-field approximation, in which correlators of spin variables taken at different points are decoupled to give a product of correlators. Taking into account that the correlator containing two variables S and two ψ at the same point is identically equal to zero, we obtain

$$M_s=\frac{2e}{i\hbar c}\sum_{ij}\Big\{\int dt\,dt_1\frac{1}{2T}\overline{x_iy_jJ_{ij}(t)J_{ij}(t_1)}\,[D^*(t,t_1)G^*(t,t_1)+D(t,t_1)G(t,t_1)]$$
$$+\sum_{kl}\int dt\,dt_1\,dt_2\,dt_3\frac{1}{(2T)^3}\overline{x_iy_jJ_{ij}(t)J_{jk}(t_1)J_{kl}(t_2)J_{li}(t_3)}\times[D^*(t,t_3)G^*(t,t_1)G^*(t_1,t_2)G^*(t_2,t_3)+G(t,t_3)D^*(t_3,t_2)\times G^*(t,t_1)G^*(t_1,t_2)+\text{h.c.}]+\ldots\Big\}. \quad (5.29)$$

The next terms of the series (denoted by ...) in (5.29) have the same structure as the first two written out: In them, the correlator of $2n$ matrices J_{ij} is multiplied by the sum of the convolutions of one function $D(t,t')$ and $2n-1$ functions $G(t,t')$, arranged in arbitrary order with equal coefficients.

Below we shall study only the response of the system to a slowly (in comparison with t_r) varying magnetic field. The correlator $\overline{x_iy_jJ_{ij}(t)\ldots J_{li}(t_k)}$ decays rapidly if the differences of all the times appearing in it become smaller than the characteristic time of the variation of the magnetic field. The correlation functions $D(t)$ and $G(t)$ are small at large $t\gg\Gamma^{-1}$ (see (5.24)–(5.25)); here $D(t)\gg G(t)$, and therefore the main contribution to the integration over the times in (5.29) is given by those regions in which the differences of the time arguments of the function D and of just one of the functions G are large (in comparison with t_r). In each of these regions, in the calculation of the correlator $x_iy_jJ_{ij}(t)\ldots J_{li}(t_n)$ we can neglect small time differences and set m_1+m_2 times equal to t, and the other $2n-m_1-m_2$ times equal to t':

$$\overline{\underbrace{J_{pr}(t)\ldots J_{qi}(t)}_{m_1}x_iy_j\underbrace{J_{ij}(t)\ldots J_{kl}(t)}_{m_2}\underbrace{J_{lm}(t')\ldots J_{np}(t')}_{2n-m_1-m_2}}$$
$$=J^{2n}\left[\frac{2\pi l_H^2}{L^2}\right]^{n-1}iL^2\xi\left[\frac{eL^2}{2\hbar c}(H(t)-H(t'))\right]\times\frac{(-1)^{m_1}-(-1)^{m_2}}{2}; \quad (5.30)$$

$$\xi(x)=\frac{1}{4}\left[\frac{\sin x}{x^2}-\frac{1}{x^2}\int_0^x\frac{\sin y}{y}dy\right].$$

After this the integral over these time differences in each of the regions can be calculated using the relation

$$\int G(t,t')\,dt'=1$$

(above T_c this relation is exact, and below T_c it is correct in leading order in $T_c - T$). Summing then the resulting series, we shall have

$$M_s = -\frac{4e}{\hbar c}\frac{NL^2T}{1+\varkappa}\left(\frac{J^2N}{4T^2},\frac{1+\varkappa}{1-\varkappa^2}\right)\int dt'[D(t,t')G(t,t') + \text{h.c.}] \times \xi\left(\frac{eL^2}{2\hbar c}(H(t)-H(t'))\right). \qquad (5.31)$$

Near the transition point the expression in round brackets in (5.31) is equal to 1, and the characteristic variations of the field H are also small: $(eL^2/2\hbar c)\Delta H \ll 1$ (see Sec. 6); therefore, the function $\psi(x)$ can be replaced by the first term of its series expansion:

$$M_s = -\left(\frac{NL^2}{12}\right)^2\frac{1}{c^2Rt_r}\frac{2}{1+\varkappa}\int dt'[D(t,t')G(t,t') + \text{h.c.}] \times (H(t)-H(t')). \qquad (5.32)$$

The formula (5.32) is valid when the characteristic time scale of the variation of the field $H(t)$ is much greater than Γ^{-1}. The expression for the linear response is obtained from (5.32) if for $D(t,t')$ and $G(t,t')$ we substitute the correlation functions $D(t-t')$ and $G(t-t')$ calculated in the constant field H_0.

Using (5.24)–(5.25), we obtain an expression for the superconducting response to a weak alternating field $H_1(t) = H_\omega e^{-i\omega t}$ with frequency $\omega \ll \Gamma$:

$$M_s(\omega) = -\left(\frac{NL^2}{12}\right)^2\frac{H_\omega}{c^2Rt_r}\frac{4}{1+\varkappa}\frac{\Gamma}{2\pi}\left\{\int_0^\infty\frac{dt}{t^2}(1-e^{i\omega t})e^{-2t/t_0} + \left(\frac{\pi}{t_0}\right)^{1/2}\int_0^\infty(\text{erf}((t/t_0)^{1/2})-1)(1-e^{i\omega t})e^{-t/t_0}\frac{dt}{t^{3/2}}\right\}. \qquad (5.33)$$

The first integral in (5.33) diverges logarithmically at small t. This formal divergence is connected with the inapplicability of formulas (5.24)–(5.25) for $t \lesssim \Gamma^{-1}$. In fact, the integral must be cut off at $t \sim \Gamma$; the second integral can be calculated exactly.[14] Taking into account also the normal contribution (5.26) and going over to the original units of time, we obtain

$$M_\omega = \frac{\mathscr{E}_\omega}{cR}\left(\frac{NL}{12}\right)^2\left\{1 + \frac{2\Gamma}{\pi(1+\varkappa)}\left[\ln\frac{C\Gamma^2\tau^{-2}}{2-i\omega t_1} + \frac{4}{i\omega t_1}\left((1-i\omega t_1)^{1/2}\,\text{arctg}(1-i\omega t_1)^{1/2} - \frac{\pi}{4}\right)\right]\right\}, \qquad (5.34)$$

where $t_1 = t_r\Gamma\tau^{-2}$ with Γ defined in (5.23), C is a constant of order unity, and $\mathscr{E}_\omega = i\omega H_\omega L^2/c$ is a Fourier component of the *emf* induced in the system by the alternating magnetic field. The expression (5.34) is applicable for

$$\omega, t_1^{-1} \ll \Gamma t_r^{-1} \approx \frac{1}{N^{1/2}}\frac{J}{\hbar}\frac{e^2R}{\hbar}. \qquad (5.35)$$

In the limit $\omega t_1 \gg 1$ we obtain

$$M_\omega = \frac{\mathscr{E}_\omega}{cR}\left(\frac{NL}{12}\right)^2\left\{1 + \frac{2\Gamma}{\pi(1+\varkappa)}\ln\frac{C\Gamma t_r^{-1}}{\omega} + \frac{i\Gamma}{1+\varkappa}\right\}. \qquad (5.36)$$

Thus, the effective conductivity of the system increases logarithmically with decrease of the frequency; the last term of (5.36) corresponds to an effective inductance $\mathscr{L} \propto \omega^{-1}$. In the opposite limit $\omega t_1 \ll 1$ we have

$$M_\omega = \frac{\mathscr{E}_\omega}{cR}\left(\frac{NL}{12}\right)^2\left\{1 + \frac{2\Gamma}{\pi(1+\varkappa)}\left[\ln\frac{C\Gamma^2}{2\tau^2} - 1 - \frac{\pi}{2}\right] + i\omega t_1\frac{4-\pi}{4\pi}\frac{\Gamma}{1+\varkappa}\right\}. \qquad (5.37)$$

In this case the conductivity increases logarithmically with decrease of $\tau = (T - T_c)/T_c$, and the inductance is proportional to the largest relaxation time; $\mathscr{L} \propto t_1$. The scale of these effects is determined by the magnitude of Γ, which is close to unity for $H \gtrsim H_0$ and small for $H \ll H_0$ (see (5.23)).

6. STATICS AND SLOW DYNAMICS OF THE MODEL BELOW THE TRANSITION POINT

Below the transition point certain relaxation times become infinite, and the system is "locked" in one of its metastable states, from which it can escape only by overcoming a thermodynamically large energy barrier. Which particular metastable state the system is found in depends on the history of the system; i.e., even for an adiabatically slow variation of the parameters (e.g., T, H) the final state of the system depends on the specific path along which the system has moved in the space of the parameters, but does not depend on the rate of motion along this path. For this the rate of motion should be small, but still larger than the exponentially (in the quantity n) small rate of the transition through the barriers. Small deviations from equilibrium in the vicinity of one metastable state relax in a time of the order of $t_1 = t_r\Gamma\tau^{-2}$ (see (5.5) and (5.23)). We shall study only the slow dynamics of the system, i.e., its response to a change of the external parameters that is slow in comparison with t_1; here we shall neglect the influence of fast-relaxation processes. In view of this it will be convenient to decompose the correlation function $D(t,t')$ and the response function $G(t,t')$ (5.11) into two parts—a fast and a slow part:

$$G^*(t,t') = \langle S^*(t)\psi(t')S(t')\rangle,$$
$$G(t,t') = -\langle S(t)\psi(t')S^*(t')\rangle = \tilde{G}(t-t') + \Delta(t,t'), \qquad (6.1)$$
$$D(t,t') = \tilde{D}(t-t') + q(t,t') = \langle S^*(t)S(t')\rangle,$$

where the "fast" functions $\tilde{G}(t)$ and $\tilde{D}(t)$ have a power-law decay for $t \gg t_r/\Gamma$, and the "slow" functions Δ and q vary over times of the order of the characteristic time t_p of the variation of the parameters of the system, which is much greater than t_r/Γ. The "fast" functions $\tilde{G}(t)$ and $\tilde{D}(t)$ satisfy the fluctuation-dissipation theorem (5.12). Adiabatic slowness of the variation of the parameters makes it possible to obtain a closed system of equations relating $q(t,t')$ and $\Delta(t,t')$ to the entire history of the variation of the parameters. We turn now to the derivation of these equations. For the calculation of $G(t,t')$ and $D(t,t')$ we shall use, as in the preceding section, the functional-integral formalism. We obtain, e.g., for $D(t,t')$

$$D(t,t) = \overline{\langle S_i^*(t)S_i(t')\rangle} = \int D\psi D\varphi\,\overline{S_i^*(t)S_i(t')\exp(A_0+A_1)},$$
$$A_0 = \int dt\sum_i(i\psi_i\dot{\varphi}_i - \psi_i^2),$$
$$A_1 = -\frac{1}{2}\int dt\sum_{i,j}(\psi_i - \psi_j)J_{ij}S_iS_j^*. \qquad (6.2)$$

Here and below we have included the factor $1/T$ in the definition of J_{ij}. After averaging over the coordinates x_i and y_j we obtain the generating functional for the correlators $D(t,t')$ and $G(t,t')$:

$$D(t,t')=\int \mathscr{D}\psi\, \mathscr{D}\varphi S_i^*(t) S_i(t') \exp(A_0+A_{eff}),$$

$$A_{eff}=\frac{1}{2}\sum_{i,j}\int dt\,dt' \frac{1}{4}\overline{J_{ij}(t)J_{ij}^*(t')}$$

$$\times[(\psi_i-\psi_j)S_iS_j^*]_t[(\psi_j-\psi_i)S_jS_i^*]_{t'} \quad (6.3)$$

$$+\frac{1}{4}\sum_{i,j,k,l}\int dt_1\,dt_2\,dt_3\,dt\frac{1}{16}\overline{J_{ij}(t)\ldots J_{ij}(t_3)}[(\psi_i-\psi_j)S_iS_j^*]_t$$

$$\ldots[(\psi_l-\psi_i)S_lS_i^*]_{t_3}+\ldots$$

The condition $N\gg 1$ enables us to make use of the mean-field approximation, in which, in the calculation of a one-point correlator consisting of variables at a point i, the factors appearing in (4.3) and pertaining to other points can be replaced by their averaged values:

$$A_{eff}=\sum_i\int dt\,dt'[-S_i(t)S_i^*(t')\psi_i(t)\psi_i(t')Q(t,t')$$

$$+(S_i(t)\psi_i(t)R^*(t,t')S_i^*(t')-\text{h.c.}], \quad (6.4)$$

where

$$Q(t,t')=\frac{1}{4}\sum_j\overline{J_{ij}(t)J_{ji}(t')}D(t,t')$$

$$+\frac{1}{16}\sum_{j,k,l}\int dt_1\,dt_2\overline{J_{ij}(t)J_{jk}(t_1)J_{kl}(t_2)J_{li}(t')}$$

$$\times[G^*(t,t_1)G^*(t_1,t_2)D(t_2,t')$$

$$+G^*(t,t_1)D(t_1,t_2)G(t',t_2)+D(t,t_1)G(t_2,t_1)G(t',t_2)]+\ldots$$

$$R(t,t')=\frac{1}{4}\sum_j\overline{J_{ij}(t)J_{ji}(t')}G(t,t') \quad (6.5)$$

$$+\frac{1}{16}\sum_{j,k,l}\int dt_1\,dt_2\overline{J_{ij}(t)J_{jk}(t_1)J_{kl}(t_2)J_{li}(t')}$$

$$\times G(t,t_1)G(t_1,t_2)G(t_2,t')+\ldots.$$

In deriving (6.4)–(6.5) we have made use of the fact that the one-point correlator $\langle\psi_i(t)\psi_i(t')S_i(t)S_i^*(t')\rangle$ can be obtained by differentiating a quantity identically equal to 1, and so is equal to 0:

$$\langle\psi_i(t)\psi_i(t')S_i(t)S_i^*(t')\rangle=\frac{\delta^2}{\delta h_t^*\,\delta h_{t'}}\int\mathscr{D}\psi\,\mathscr{D}\varphi$$

$$\times\exp\left[A-\int\sum_i\psi_i(hS_i^*-h^*S_i)dt\right].$$

We resolve the functions $Q(t,t')$ and $R(t,t')$ into a fast and a slow part:

$$Q(t,t')=\tilde{Q}(t,t')+Q_0(t,t'),\quad R(t,t')=\tilde{R}(t,t')+R_0(t,t').$$

The functions $D(t,t')$ and $G(t,t')$ each consist of a fast and a slow part (formula (6.1)), and the terms in $Q(t,t')$ and $R(t,t')$ containing only the fast parts $\tilde{D}(t,t')$ and $\tilde{G}(t,t')$ lead to the rapidly decaying functions $\tilde{Q}(t,t')$ and $\tilde{R}(t,t')$. The part of the effective action A_{eff} due to the functions $\tilde{Q}(t,t')$ and $\tilde{R}(t,t')$ leads to a renormalization of the magnitude of the thermal noise and of the fast-relaxation law. Since in times of the order of t_1 the external parameters of the system change slowly, over these times thermal equilibrium can be established within the given metastable state. For the slow dynamics, only the slow parts of the functions $Q(t,t')$ and $R(t,t')$, containing at least one factor $\Delta(t,t')$ or $q(t,t')$, are important.

Below we shall confine ourselves to studying the system near the transition point T_c, where both functions $q(t,t')$ and $\Delta(t,t')$ are small and, as will be shown below, $q\sim\tau$ and $\Delta\sim\tau^2$. Throughout this part, $\tau=(T_c-T)/T_c>0$ below the transition point. We can also assume that the changes of the external parameters are small ($\tau\ll 1$ and $e\Delta HL^2/\hbar c\ll 1$) and take them into account only in terms of lowest order in q and Δ in formula (6.5) for $Q(t,t')$ and $R(t,t')$. Only one of the slow functions (q or Δ) appears in these terms, and therefore the arguments of J_{ij} in the correlator $\overline{J...J}$ in formula (6.5) coincide either with t or with t'. We shall calculate such correlators. Just as in the derivation of formula (2.4), we obtain

$$\overline{\underbrace{J_{ij}(t)\ldots J_{kl}(t)}_{m}\underbrace{J_{lr}(t')\ldots J_{pi}(t')}_{2n-m}}$$

$$=\frac{J^{2n}}{T^m(t)T^{2n-m}(t')}\left(\frac{2\pi l_H^2}{L^2}\right)^{n-1}$$

$$\times\begin{cases}1, & m=2k\\ 1-\dfrac{1}{2}\left[\dfrac{L^2(H(t)-H(t'))}{6\hbar c}\right]^2, & m=2k+1.\end{cases} \quad (6.6)$$

In the next terms of the expansion of (6.5) in powers of $q(t,t')$ and $\Delta(t,t')$ we can assume that the external parameters are constant and set the temperature equal to T_c In formula (6.5) the time integration of the fast parts $\tilde{G}(t,t')$ of the response functions can be performed independently and between infinite limits. We denote

$$g(t)=\frac{T_c}{T(t)}\int dt_1\tilde{G}(t,t_1). \quad (6.7)$$

Substituting (6.6) and (6.7) into (6.5) and collecting terms of first and second order in $\Delta(t,t')$, we obtain an expression for $R_0(t,t')$ (the slow part of $R(t,t')$):

$$R_0(t_1,t_2)=\mathscr{M}(t_1,t_2)\Delta(t_1,t_2)+\eta\int dt'\Delta(t_1,t')\Delta(t',t_2), \quad (6.8)$$

$$\mathscr{M}(t_1,t_2)=\frac{J^2N}{4T_1T_2}\left\{1-\frac{1}{2}\left[\frac{eL^2(H_1-H_2)}{6\hbar c}\right]^2\right.$$

$$\left.+\varkappa g_1g_2\right\}\frac{1}{(1-\varkappa g_1^2)(1-\varkappa g_2^2)},$$

$$\eta=\frac{J^2N}{4T_c^2}\frac{\varkappa(3+\varkappa)}{(1-\varkappa)^3}=\frac{\varkappa(3+\varkappa)}{1-\varkappa^2}.$$

where $g_{1,2}=g(t_{1,2})$ and $T_{1,2}=T(t_{1,2})$. In the calculation of $Q_0(t,t')$ it is sufficient to confine ourselves to terms of first order in $\Delta(t,t')$; as a result we obtain

$$Q_0(t_1,t_2)=\mathscr{M}(t_1,t_2)q(t_1,t_2)$$

$$-\eta\int dt[\Delta^*(t_1,t)(q(t,t_2)+\tilde{D}(t,t_2))$$

$$+(q(t_1,t)+\tilde{D}(t_1,t))\Delta(t_2,t)]. \quad (6.9)$$

The integral $\int dt'\Delta(t,t')$ describes the response of the system to a time-independent perturbation, and therefore the quantity $\Delta(t,t')$ is inversely proportional to the time scale of the variation of the external parameters of the system; this physical argument will be confirmed below. Taking into account that at large times $\tilde{D}(t)$ decays like $t^{-1/2}$, we find that the term in (6.9) containing $\tilde{D}(t)$ can be neglected if the characteristic time t_p of the variation of the external parameters of the system satisfies $t_p \gg t_1$, which we shall assume to be the case in the following.

The second term in the effective action (6.4) has the same form as the action for a spin S_i situated in an external field

$$h_i^*(t)=\int dt' R^*(t,t')S^*(t').$$

In order to bring the first term ($A_{\rm eff}^{(1)}$) in $A_{\rm eff}$ to the same form, we introduce an auxiliary Gaussian field $z(t)$ with correlator $Q(t,t')$:

$$\exp A_{eff}^{(1)}=\left[\exp\left\{\int dt\, S(t)\psi(t)\tilde{h}^*(t)-\text{h.c.}\right\}\right]_z, \qquad (6.10)$$
$$\tilde{h}(t)=z(t)+\int dt'\, R(t,t')S(t'),$$

where the square brackets denote averaging over $z(t)$:

$$[C(z)]_z\equiv\frac{\int \mathcal{D}z\mathcal{D}z^* C(z)\exp\left\{-\int z^*(t)Q^{-1}(t,t')z(t')\,dt\,dt'\right\}}{\int \mathcal{D}z\mathcal{D}z^*\exp\left\{-\int z^*(t)Q^{-1}(t,t')z(t')\,dt\,dt'\right\}}. \qquad (6.11)$$

To calculate the correlators $D(t,t')$ and $G(t,t')$ with the action (6.10)–(6.11) we change the order of integration over $z(t)$ and $S(t)$, average first over the fast thermal fluctuations (i.e., integrate over $\psi(t)$ and $S(t)$), and then average the resulting expression for the correlators over the slow field $z(t)$.

The characteristic times of the variation of the field $\tilde{h}(t)$ are the same as those of the variation of the external parameters of the system; therefore, over the time $\sim t_1$ needed for equilibrium to be established $\tilde{h}(t)$ can be assumed to be constant and the average value $\mu(t)$ of the spin $S(t)$ can be determined from the static formulas

$$\mu(t)\equiv\langle S(t)\rangle=\frac{I_1(2|\tilde{h}(t)|)}{I_0(2|\tilde{h}(t)|)}\frac{\tilde{h}(t)}{|\tilde{h}(t)|}$$
$$\approx\tilde{h}(t)\left[1-\frac{|\tilde{h}(t)|^2}{2}+\frac{1}{3}|\tilde{h}(t)|^4\right]. \qquad (6.12)$$

Near the transition point the characteristic magnitudes of $\tilde{h}(t)$ are small: $[\tilde{h}^2]_z^2\sim q\sim\tau$, and it was this which enabled us to confine ourselves to expanding $\mu(t)$ to terms of fourth order in $\tilde{h}(t)$ in formula (6.12).

The slowness of the function $R_0(t,t')$ makes it possible to replace $S^*(t')$ in the integral $\int dt' R^*(t,t')S^*(t')$ by its average value $\mu^*(t')$. We obtain

$$\tilde{h}(t)=z(t)+\int R_0(t,t')\mu(t')\,dt'+h(t). \qquad (6.13)$$

In the expression (6.13) for $\tilde{h}(t)$ we have included the field $h(t)$ conjugate to the spin variable $S(t)$, in such a way that the response function $G(t,t')$ can be obtained by differentiating $[\langle S(t)\rangle]_z$ with respect to $h(t')$:

$$G(t,t')=\frac{\partial[\langle S(t)\rangle]_z}{\partial h(t')}=\left[\frac{\partial\mu(t)}{\partial h(t')}\right]_z. \qquad (6.14)$$

Equations (6.12) and (6.13) form a nonlinear integral equation for $\mu(t)$ in terms of $z(t)$. This equation can be solved by iterations in the small quantity $R_0(t,t')$. In the expression for the response it is sufficient to retain just terms of first and second order in $R_0(t,t')$, and in the coefficient in the term of second order in $R_0(t,t')$ we can confine ourselves to lowest order in $|\tilde{h}(t)|^2$. For the slow part of the response function we obtain

$$\Delta(t,t')=\Big[(1-|z|^2+|z|^4)_t(1-|z|^2+|z|^4)_{t'}R_0(t,t')$$
$$+\left(-\frac{1}{2}z^2+\frac{2}{3}z^2|z|^2\right)_t$$
$$\times\left(-\frac{(z^*)^2}{2}+\frac{2}{3}(z^*)^2|z|^2\right)_{t'}R_0^*(t,t')\Big]_z$$
$$+\int dt'' R_0(t,t'')\Delta(t'',t'). \qquad (6.15)$$

Averaging over $z(t)$ with the correlator $Q_0(t,t')$, we obtain

$$\Delta(t,t')=[1-Q_0(t,t)-Q_0(t',t')+2Q_0^2(t,t)+2Q_0^2(t',t')$$
$$+|Q_0(t,t')|^2+Q_0(t,t)Q_0(t',t')]R_0(t,t')$$
$$+\frac{1}{2}Q_0^2(t',t)R_0^*(t,t')$$
$$+\int dt'' R_0(t,t'')\Delta(t'',t'). \qquad (6.16)$$

We shall also calculate the slow part $q(t,t')$ of the correlation function by first averaging over the fast thermal fluctuations in a fixed field z and then averaging over z. The averaging over the thermal fluctuations at the times t and t' can be performed independently, and therefore

$$q(t,t')=[\langle S^*(t)S(t')\rangle]_z=[\mu^*(t)\mu(t')]_z. \qquad (6.17)$$

Expressing $\mu(t)$ in terms of $z(t)$ and $R_0(t,t')$ by means of the formulas (6.12) and (6.13), and retaining only terms of first order in R_0, we obtain

$$q(t,t')=Q_0(t,t')\Big[1-Q_0(t,t)-Q_0(t',t')+2Q_0^2(t,t)$$
$$+2Q_0^2(t',t')+Q_0(t,t)Q_0(t',t')+\frac{1}{2}|Q_0(t,t')|^2\Big]$$
$$+\int dt'' R_0^*(t,t'')Q_0(t'',t')+\int dt'' Q_0(t,t'')R_0(t',t''). \qquad (6.18)$$

The equations (6.16) and (6.18) together with the formulas (6.8) and (6.9) form a closed system of nonlinear integral equations determining $q(t,t')$ and $\Delta(t,t')$. The functions $g(t)$ appearing in the quantity R_0 (formula (6.8)) can be expressed in terms of $q(t,t)$ by means of the fluctuation-dissipation theorem, which is valid for the fast parts of the correlation functions:

$$g(t)=\frac{T_c}{T(t)}\int dt'\,\tilde{G}(t,t')=\frac{T_c}{T(t)}\tilde{D}(t,t)$$
$$=\frac{T_c}{T(t)}(D(t,t)-q(t,t))$$
$$=\frac{T_c}{T(t)}(1-q(t,t)). \qquad (6.19)$$

In the order in τ under consideration, $q(t,t)$ and $g(t)$ do not depend on the history of the system and can be found without a complete solution of the nonlinear integral equations (6.18), (6.16), (6.8), and (6.9). For this we consider the particular case of these equations when $t \to t'$. The functions $R_0(t,t')$ and $\Delta(t,t')$ are retarded, and therefore in this limit integral terms of the type $\int dt''\Delta(t,t'')\Delta(t'',t')$ can be omitted. In the leading approximation in τ, $\mathscr{M}(t,t') = 1$, $R_0(t,t') = \Delta(t,t')$, and $Q_0(t,t') = q(t,t')$; therefore, using these relations in the terms of leading order in $\tau_t = \tau(t)$, we obtain

$$R_0(t,t) = \frac{1}{(1-\tau_t)^2} \frac{1+\varkappa g^2(t)}{(1-\varkappa g^2(t))^2} \frac{(1-\varkappa)^2}{1+\varkappa} \Delta(t,t), \quad (6.20a)$$

$$Q_0(t,t) = (1-\tau_t)^{-2} \frac{1+\varkappa g^2(t)}{(1-\varkappa g^2(t))^2} \frac{(1-\varkappa)^2}{1+\varkappa} q(t,t)$$
$$+\eta \int dt' [\Delta^*(t,t')q(t',t) + q(t,t')\Delta(t,t')]. \quad (6.20b)$$

$$\Delta(t,t) = [1-2Q_0(t,t) + 6q^2(t,t)] R_0(t,t) + \frac{1}{2} q^2(t,t)\Delta^*(t,t). \quad (6.20c)$$

$$q(t,t) = \left[1-2Q_0(t,t) + \frac{11}{2} q^2(t,t)\right] Q_0(t,t)$$
$$+\int dt' [\Delta^*(t,t')q(t',t) + q(t,t')\Delta(t,t')]. \quad (6.20d)$$

The system of equations (6.20) admits the trivial solution with $\Delta(t,t') = R_0(t,t') = 0$. This solution coincides with the static solution obtained in the replica method with unbroken replica symmetry (for the analogous result in the case $H \gg H_0$ see Refs. 1 and 15), is unstable, and leads to nonphysical results at low temperatures. The system of equations (6.20) also admits another solution, with $\Delta(t,t') \neq 0$. Unlike the solution with $\Delta(t,t') = 0$, this solution leads to violation of the fluctuation-dissipation theorem for the slow part of the response. The equations (6.20a) and (6.20c) form a linear homogeneous system of equations for the quantities $R_0(t,t')$ and $\Delta(t,t')$. The compatibility condition for this system reduces to an algebraic equation for $Q_0(t,t)$ and $q(t,t)$, which, together with Eqs. (6.20b) and (6.20d), forms a closed algebraic system of equations for $q(t,t)$, $Q_0(t,t)$, and the magnitude of the integral term in the right-hand side of (6.20b) and (6.20d). Solving this system in the required order in τ, we obtain

$$q(t,t) = \tau_t + \frac{3}{4(1+\eta)}\tau_t^2, \qquad Q_0(t,t) = \tau_t + [3/4(1+\eta)+2]\tau_t^2, \quad (6.21)$$

$$P = \int dt' [\Delta^*(t,t')q^*(t,t') + q(t,t')\Delta(t,t')] = (1+\eta)^{-1}\tau_t^3.$$

The Edwards-Anderson order parameter is the limit

$$q_{EA} = \lim_{t\to\infty} \langle S^*(0)S(t)\rangle,$$

where by $t \to \infty$ we mean times large compared with t_1 but small compared with the times of passage through the barriers separating different metastable states and small compared with the characteristic time t_p of the variation of the external parameters; therefore, q_{EA} coincides with $q(t,t)$ (see (6.21)). In the order in τ under consideration, q_{EA} does not depend on the history of the system and coincides with the result of the theory of Parisi[16] ($q_{EA} = q(x=1)$ in the theory of Parisi) and Sompolinsky[17] for two-component spins.[18]

Having substituted $q(t,t)$ and $Q_0(t,t)$ from (6.21) into the system of equations (6.16), (6.18), (6.8), (6.9), we retain in these equations only the terms of leading order in τ_t and make the change of variable $\bar\Delta = 4(1+\eta)\Delta/3$. We obtain

$$\left\{\frac{4}{3}|q(t_1,t_2)|^2 - \frac{1}{2}\left[\frac{eL^2(H_1-H_2)}{6\hbar c}\right]^2 - (\tau_1^2+\tau_2^2)\right\}\bar\Delta(t_1,t_2)$$
$$+\frac{2}{3}q^2(t_2,t_1)\bar\Delta^*(t_1,t_2) + \int dt\, \bar\Delta(t_1,t)\bar\Delta(t,t_2) = 0, \quad (6.22)$$
$$\left\{\frac{2}{3}|q(t_1,t_2)|^2 - \frac{1}{2}\left[\frac{eL^2(H_1-H_2)}{6\hbar c}\right]^2 - (\tau_1^2+\tau_2^2)\right\}q(t_1,t_2)$$
$$+\int dt [\bar\Delta^*(t_1,t)q(t,t_2) + \bar\Delta(t_2,t)q^*(t,t_1)],$$

where $\tau_{1,2} \equiv \tau(t_{1,2})$. The form of Eqs. (6.22) does not depend on the magnitude of the constant magnetic field, which determines only the scale of the quantity $\Delta(t_1,t_2)$ (the dependence $\eta(H)$ is determined from the formulas (6.9), (4.6), and (4.7)). The equations (6.22) admit a purely real solution $\bar\Delta = \bar\Delta^*$, $q = q^*$. The solutions of the system of equations (6.22) for a constant field H but different $\tau(t)$ (i.e., different rates of cooling) can be obtained from one solution

$$\Delta_p(t,t'), \quad q_p(t,t')$$

with $\tau = t$. If $\tau(t)$ is a monotonic function, then

$$q(t,t') = q_p(\tau(t),\tau(t'))$$

and

$$\Delta(t,t') = \Delta_p(\tau(t),\tau(t'))\frac{d\tau}{dt'}$$

satisfy Eq. (6.22). If $\tau(t)$ is a nonmonotonic function, then

$$q(t,t') = q_p(\tau(t),\tau(t'))$$

as before, but the expression for $\Delta(t,t')$ turns out to be more complicated; namely, $\Delta(t,t') = 0$ for all t' such that there exists a $t'' > t'$ such that $\tau(t') \geqslant \tau(t'')$ (see Fig. 4); for all other values of t' the expression for $\Delta(t,t')$ is unchanged. In the spin-glass theory of Refs. 2 and 17 the difference between the spin-glass susceptibilities measured by the method of cooling in a field and without a field was investigated; in our notation this quantity is equal to $\int dt'\Delta(t,t')$. The quantity $\int dt'\Delta(t,t')$ does not depend on the cooling process (i.e., on the form of $\tau(t')$ with $\tau(t) = \tau_0$) and is equal to $3\tau_0^2/4$ (the coefficient of τ_0^2 was obtained from the solution of the system of equations (6.22) with $\tau = t$), which coincides with the result obtained by the replica method.[18]

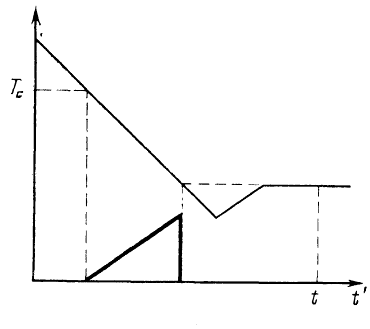

FIG. 4. Example of the dependence of the anomalous response $\Delta(t,t')$ (shown by the thick line) for a nonmonotonic variation of the temperature $\tau(t')$ (depicted by the thin line).

In the system studied here the physically measurable quantity is the magnetic moment of the currents that are induced by the magnetic field. We shall consider the most natural situation, when the magnetic field changes by ΔH after the end of the cooling process. We shall assume that the cooling process ended at time $\tilde{t}_1$, the field began to vary slowly at time t_2, had changed by ΔH by time t_3, and did not change further, and the measurement of the induced magnetic moment was performed at a later time t: $t > t_3 > t_2 > \tilde{t}_1$. The magnetic moment (at the time t) is given by the formula (5.32). In the situation under consideration the factor $(H(t) - H(t'))$ in the integral (5.32) vanishes for $t' \geqslant t_3$, and for $t' \leqslant t_2$ is constant and equal to ΔH; therefore, a contribution to the integration over the time in (5.32) is given only by the region $t' \leqslant t_3$, in which the functions $D(t,t')$ and $G(t,t')$ can be replaced by their slow parts $q(t,t')$ and $\Delta(t,t')$. Moreover, in the approximation linear in ΔH we can use $q(t,t')$ and $\Delta(t,t')$ calculated for the situation without change of the field, for which $\Delta(t,t')$ vanishes for $t' > \tilde{t}_1$; therefore, in (5.32) the integral of the product of the function $q(t,t')$ and $\Delta(t,t')$ coincides with the integral P defined in formula (6.21). (When the time dependence of the field in Eqs. (6.22) for $q(t,t')$ and $\Delta(t,t')$ is taken into account this statement becomes incorrect, since $\Delta(t,t') \neq 0$ for $t_3 > t' > t_2$.) This gives the possibility of obtaining an answer for the linear response without solving the full system of equations (6.22):

$$M_s = -\left(\frac{NL^2}{12}\right)^2 \frac{2}{c^2 R t_r} \frac{1-\varkappa}{1+3\varkappa} \tau^3 \Delta H, \tag{6.23}$$

where the parameter $\varkappa$ is given by (4.6) and (4.7). Expressing the quantities in (6.23) in terms of the original parameters of the problem, we have, finally,

$$M_s = -\left(\frac{NL}{12}\right)^2 \frac{e^2}{\hbar c^2} \frac{J}{\hbar N^{1/2}} \Delta H L^2 B \tau^3, \qquad B = \frac{4(1+\varkappa)^{1/2}}{1+3\varkappa}. \tag{6.24}$$

In the limits of strong and weak fields, $\varkappa$ is equal to 0 and 1, respectively; therefore, in these limiting cases B takes the values $B = 4$ for $H \gg H_0$ and $B = 2^{1/2}$ for $H \ll H_0$. The response (6.24) is diamagnetic, as in ordinary superconductors.

The linear-response approximation is valid as long as the effect of the term contain ΔH in Eq. (6.22) on the solution $q(t,t')$ and $\Delta(t,t')$ can be neglected i.e., as long as

$$eL^2 \Delta H / 6\hbar c \ll \tau.$$

For a larger change of the field H ($\Delta H_{\max} \sim 6\hbar c \tau / eL^2$), the superconducting magnetic moment reaches its maximum value. This can occur either smoothly with increase of ΔH ($M_s \to M_{s\,\max}$), or discontinuously when the adiabatic solution of Eqs. (6.22) becomes unstable. The question of which of these two scenarios is realized requires a special analysis. In any case, an estimate of the quantity $M_{s\,\max}$ can be obtained by substitution of the estimate for $\Delta H_{\max}$ into formula (6.24):

$$M_{s\,max} \approx 6\left(\frac{NL}{12}\right)^2 \frac{e}{c} \frac{J}{\hbar N^{1/2}} B \tau^4. \tag{6.25}$$

7. CONCLUSION

1. We have shown that a system of irregularly arranged Josephson junctions, placed in a not too weak magnetic field (such that the flux of the field through the entire system is much greater than a flux quantum—see (2.1)), undergoes a phase transition to a macroscopically coherent disordered state. This state is the superconducting analog of a spin glass with two-component XY spins. The phase-transition temperature T_c is given by the formulas (4.7)–(4.9) and Fig. 3. As the temperature T approaches the transition point T_c from above, critical slowing-down of the fluctuations of the phases of the individual filaments occurs, and this is manifested in the appearance of an anomalous part in the response of the system to a weak alternating magnetic field (see (5.34)–(5.37)). The effective conductivity grows logarithmically with decrease of the field frequency ω and of the quantity $\tau = T/T_c - 1$ representing the distance to the transition point. In addition, the response is characterized by the appearance of an effective inductance $\mathscr{L} \propto \min(\omega^{-1}, t_1)$, where

$$t_1 \sim \hbar^2 N^{1/2} \tau^{-2} / c^2 R J \tag{7.1}$$

is the maximum relaxation time (see (5.34)). These effects do not contain any small factor in strong fields $H \gtrsim H_0$ (the field H_0 is defined by the condition $eH_0 Ll/\pi\hbar c = 1$ (see (3.8)), and are proportional to $(H/H_0)^{1/2}$ for $H \ll H_0$ (in contrast to ordinary superconductors, in which these effects are small in the Ginzburg-Levanyuk parameter $\mathrm{Gi} \ll 1$). The characteristic scale of the frequencies ω and relaxation times t_1 at which these effects should be observed is determined by the formula (5.35) and does not depend on the relative magnitude of H and H_0; the corresponding temperature interval is small for $H \ll H_0$: $\tau \lesssim (H/H_0)^{1/2}$.

At temperatures $T < T_c$ the system possesses a diamagnetic response to a quasistatic change of the magnetic field (i.e., a change with a characteristic time scale t_p much smaller than the exponentially large (in N) times of transitions between different metastable states); see (6.24). An analogous result for vector spin glasses was obtained in Ref. 19, in which it was shown by the replica method in the mean-field approximation that the transverse spin stiffness $\rho_s \sim \tau^3$. The diamagnetic contribution to M_s is much greater than the dynamical contribution (5.34) for a sufficiently slow variation of the field:

$$\frac{1}{t_p} \ll \frac{e^2 R}{\hbar} \frac{J}{\hbar N^{1/2}} |\tau|^3. \tag{7.2}$$

An estimate for the maximum magnitude of M_s (the analog of the critical current in superconductivity) has been obtained in (6.25).

It should be noted that although the ratio H/H_0 does not appear explicitly in the formulas (6.24)–(6.25) and (7.2), the applicability of these formulas for $H \ll H_0$ is limited by the condition

$$|\tau| \lesssim \tau_H = (H/H_0)^{1/2} \ll 1, \tag{7.3}$$

inasmuch as the expansion parameter near the transition point for $H \ll H_0$ is essentially $|\tau|/\tau_H$ (taking (6.19) into account, we find that the expansion in q in (6.20a) and (6.20c) is valid for $q \lesssim 1 - \varkappa_c$, which, for $H \ll H_0$, is equivalent to (7.3)). The properties of the low-temperature phase for $|\tau| \gtrsim \tau_H$ require a separate investigation; it is not ruled out that a first-order phase transition occurs at $|\tau| = \tau_H^* \sim \tau_H$; the corresponding transition line is shown qualitatively by the dashed curve in Fig. 3. Another possibility is a crossover at $\tau \sim \tau_H$ to a state with the same symmetry but characterized by different dependences of the physical quantities on τ.

In this paper we have considered the situation in which the transition temperature T_c of the system is much lower than the temperature T_0 of the superconducting transition in an individual strip, so that the magnitude of the Josephson coupling J does not depend on the temperature. This restriction is not fundamental; the qualitative picture is also preserved for $T_c \cong T_0$, but the specific temperature dependences are changed, since one has to take into account the dependence of J on $T_0 - T$. In the leading approximation in τ we must assume that $\tau = (T - T_c)\,(T_c^{-1} - \partial \ln J/\partial T)$.

2. We have considered a rather particular example of a disordered system of microjunctions between superconducting regions, characterized by the fact that each region has microjunctions with a large number of regions $Z = N$. In the commoner physical situations this number is of order unity (e.g., in systems of superconducting balls in a dielectric matrix or in a matrix made of normal metal). The relationship between such systems and the system considered by us is the same as that between real spin glasses, with short-range interaction, and the Sherrington-Kirkpatrick model. The modeling of spin glasses on the basis of systems of superconducting microjunctions gives the possibility of studying systems with a variable number Z of links experimentally, and, thus, of studying the transition between a Sherrington-Kirkpatrick model with $Z \sim N$ and systems with short-range interaction. It may be that this will make it possible to ascertain experimentally whether a phase transition exists in an isotropic planar spin glass. We emphasize again that, unlike magnets, in which there is always at least a weak anisotropy in spin space, in superconducting systems such anisotropy is forbidden by the condition of global gauge invariance. A phase transition in an isotropic planar spin glass with finite Z was predicted theoretically in Ref. 12: however, the existing numerical experiments[20] indicate, rather, the absence of a phase transition. We suggest, however, that the system sizes achievable for the modeling are too small to permit reliable conclusions. Systems with a variable number of links can be created, e.g., from superconducting needles randomly dispersed so that each needle has microjunctions with many others but the length of a needle is much smaller than the size of the system (a "haystack").

Systems of superconducting balls with a small number of links, such that the system as a whole is near the percolation threshold, were studied in Ref. 7, in which it was shown that in the mean-field approximation the phase diagram in strong magnetic fields has a low-temperature region corresponding to a spin glass. In Ref. 7, only the equilibrium thermodynamics was considered, without allowance for breaking of the replica symmetry, and therefore the diamagnetic response effect investigated by us could not be obtained in Ref. 7.

In the system studied by us all the energy barriers between different metastable states are large in terms of the parameter specifying the total number of elements of the system. Neglecting the rare passages through such barriers, we obtained Eqs. (6.22), which describe the state of the system and its dependence on the history of the system, i.e., on the path in the (H,T) plane by which this state was reached. It is interesting, however, that the state of the system in this approximation does not depend on the shape of the $T(t)$ curve if the temperature is changed at a constant magnetic field, even if this change of temperature is nonmonotonic. The form of Eqs. (6.22) is universal—it does not depend on the absolute magnitude of the magnetic field H in the region of their applicability ($eHL^2/c\hbar \gg 1$, $\tau = (T_c - T)/T_c \ll (H/H_0)^{1/2}$). The magnitude of the Edwards-Anderson parameter, as obtained from Eqs. (6.22), does not depend on the history of the system in the order in τ considered by us, and coincides with the result[18] of the theory of Parisi and Sompolinsky. In the theory of spin glasses it is customary to study the quantity $\Delta = q(1) - q(0)$, describing the extent of the breaking of the replica symmetry of the theory. In the case of ordinary spin glasses Δ has a simple physical meaning—it is the difference between the susceptibility measured by the method of "cooling in the field" and that measured by the usual method. In a system of superconducting microjunctions Δ does not have a simple physical meaning, but its magnitude can be obtained from the solution of Eqs. (6.2): $\Delta = \int dt'\Delta(t,t')$. In the leading order in τ the quantity Δ does not depend on the manner of cooling (the shape of the curve $\tau(T)$), if this cooling is performed in a constant field, and coincides with the result of the theory of Ref. 18. The equations (6.22) describe the slow (adiabatic) evolution of the system under the influence of the slow evolution of the external parameters. It is possible that the solution of Eqs. (6.22) becomes unstable for some value of the parameters, and the system rapidly (in times $\sim t_r$) changes its state. The question of the stability of the solution of Eqs. (6.22) against such fast perturbations, for different histories of the system, requires a special analysis.

In conclusion we note a surprising analogy between the link matrix J_{ij} of our system of microjunctions and the link matrix of the associative-memory model of Hopfield[21-23]: In both models, only cyclic correlators of the quantities J_{ij} (as in formula (2.3)) are important, and the magnitudes of these correlators are given by relations of the same type.

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