Langevin description of mesoscopic fluctuations in disordered media

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A procedure, analogous to the Langevin scheme of calculating thermodynamic fluctuations, is proposed for measuring mesoscopic fluctuations of the conductance of metallic samples and of the fluctuations of the optical transparency of elastically scattering media. It is shown that the magnitude of the fluctuations of the conductance of small metallic samples, measured by a four-probe procedure, depends on the sizes of the measurement contacts, and that when these sizes are increased the fluctuations can greatly exceed $e^2/h$. The relative fluctuations of the transparency of mesoscopic samples greatly exceed the relative fluctuations of the conductance.

I. INTRODUCTION

This paper is devoted to the study of the current-density spatial fluctuations produced in disordered metals by an electric field and due to random arrangement of the impurities in the sample (mesoscopic fluctuations), and also to the analogous light-energy flux fluctuations produced in inelastically scattering media irradiated by coherent light. These fluctuations are calculated by a scheme reminiscent of the Langevin scheme of calculating thermodynamic fluctuations. This scheme can be used because the spatial fluctuations of the current density in disordered metals (or of the energy-light flux energy in turbid media) can be divided into microscopic (over scales shorter than the mean free path $l$) and diffusional (over scales longer than $l$). The microscopic current-density fluctuations, averaged over a spatial scale of order $l$, play the role of extraneous currents and diffusion equations, which describe fluctuations over diffusional scales.

The difference between the description of mesoscopic fluctuations of the conductivity in metals from that of optical transparency of elastically scattering media is due to the different definitions of their extraneous-current correlators.

In the description of mesoscopic conductivity fluctuations in disordered metals, the extraneous currents are proportional to the gradient of the electrochemical potential. In this case, the use of the Langevin scheme makes the calculations easier than in the usual diagram technique in those situations in which the electric field is not uniform (e.g., in point contacts), and in samples of complicated geometry. In particular, such a scheme permits construction of a theory of the measurement of mesoscopic fluctuations in disordered metallic systems, i.e., identification of the quantity measured in the experiment.

It is shown in Refs. 5 and 6 that conductance fluctuations in weakly disordered (i.e., metallic samples are of the order of $\langle Mg \rangle = \langle (G - (G)^2) \rangle = e^2/h$.

Here $\langle \ldots \rangle$ denotes averaging over the realizations of the random potential (i.e., over the samples), and $p_{F}$ is the Fermi momentum in the metal. The conductance $G$ was measured here by a two-probe method. In experiment, however, the conductance is most frequently measured by a four-probe method, with two contacts used to set the current flowing through the sample, and the electrochemical-potential difference picked off the other two contacts (Fig. 1).

We shall show that in this case the fluctuations of the electrochemical potential across the measuring contacts is determined by gigantically macroscopic fluctuations averaged over the contact dimensions. The quantity $\langle (\Delta G)^2 \rangle$ measured by the four-probe method should therefore depend strongly on the sizes of the measuring contacts, and that when these sizes are increased the fluctuations of $G$ can greatly exceed $e^2/h$. Finally, it is possible to measure in experiment the conductances $G_1$ and $G_2$ of the point contacts $C_1$ and $C_2$ themselves (see Fig. 1). It will be shown below that there is a correlation exists between the quantities $\Delta G_1$ and $\Delta G_2$ and falls off as a power law with increasing distance between the contacts.

Another possible use of the Langevin scheme proposed here is for investigations of mesoscopic fluctuations of the optical transparency of elastically scattering media. If a coherent light beam is incident on an elastically scattering medium, the intensity of the scattered light fluctuates as a function of the location of the scatterers if the scattering angle is given, or as a function of the scattering angle if the scatterer locations are given. The interference pattern resulting from the scattering is called a speckle. A fairly complete theory was developed for speckles produced as a result of single scattering.

We develop in the present paper a theory for speckles resulting from the passage of light through a multiply scattering medium. The extraneous currents are in this case proportional to the average density of the light energy at the given point (and not to the gradient of the electrochemical potential as in the case of metal-conductance fluctuations). As a result, the relative fluctuations of the optical transparency of mesoscopic samples turns out to be much larger than the relative fluctuations of the conductance of metallic samples. This demonstrates the difference between the mechanisms that give rise to extraneous currents in the two cases discussed above.

II. TRANSPARENCY FLUCTUATIONS OF DISORDERED MEDIA

a. We begin with a study of stationary spatial fluctuations of the density and of the light flux energy, which are produced when a stationary coherent electromagnetic wave having an energy flux density $J$ and a wavelength $\lambda$ is incident on the interface of an elastically scattering medium.
with \( v_{\text{free}} \). The geometric dimensions of the medium are assumed much larger than the mean free path \( l \). In the case \( l \gg \lambda \) we can find the density and flux of the light energy in the medium, averaged over the realizations of a random potential, by using the kinetic equation, whose solution shows that the energy flux \( J \) and energy density \( \rho \) vary smoothly as functions of the coordinates over scales exceeding \( l \). If \( l \ll \lambda \), the density and flux of the light energy in the medium, both averaged over the realizations of the random potential can be found by using the kinetic equation, whose solution shows that the energy flux \( J \) and energy density \( \rho \) vary smoothly with the coordinates over scales exceeding \( l \). To describe the spatial fluctuations of the energy flux in the scattering medium, averaged over scales larger than the mean free path \( l \), we use a scheme similar to Langevin’s scheme of calculating thermodynamic fluctuations.\(^1\)\(^2\) The fluctuations over small scales assume here the role of random extraneous fluxes \( J_{\text{extr}}(r) \) in the diffusion equations

\[
\begin{align*}
d\rho &= 0, \\
J &= -DVn + J_{\text{extr}}(r).
\end{align*}
\]

Here \( D = \kappa/c \) is the diffusion coefficient and \( c \) the speed of light.

Equations (1) and (2) must be supplemented by the usual boundary conditions for the diffusion equations; they describe then the spatial fluctuations of the current and coincide with the equations that describe thermodynamic fluctuations.\(^1\)\(^2\) The difference lies in the definition of the microscopic random currents \( J_{\text{extr}} \). In our case the correlator of the random extraneous currents is obtained by summing the diagrams of Fig. 2a, and at \( |r - r'| > l \) we have

\[
\langle J_{\text{extr}}(r)J_{\text{extr}}(r') \rangle = \frac{\hbar}{\pi} \delta(r - r') \delta(\rho - \rho'),
\]

Equations (1)–(3) are valid at an arbitrary sample geometry, and \( \langle n(r) \rangle \) is the solution of the diffusion equations (1) and (2) without extraneous currents \( J_{\text{extr}}(r) \).

The phenomenological scheme (1)–(3) is based on the assumption that spatial fluctuations of the energy fluxes and densities resulting from random interference of the waves scattered by the randomly distributed scatterers can be grouped into microscopic fluctuations \( |r - r'| < l \) and diffusional ones \( |r - r'| > l \). To check on this assumption we obtained all the results of this study by using the much more complicated diagram-summation procedure. To determine the transparency of an elastically scattering medium it was necessary to sum the diagrams shown in Fig. 2b, which describe electron diffusion at \( |r - r'| > l \).

The qualitative interpretation of Eq. (3) is the follow-

\[\text{FIG. 1}\]

\[\text{FIG. 2}\]
FIG. 3.

be summed to calculate \( \langle J_{\alpha}(r) \rangle \). The expression obtained agrees with the known fact that, in a speckle, the Poisson distribution functions of the microscopic density \( F(n) \) and the light energy \( P \) are given by

\[
F(n) = e^{-n} \cdot \frac{n^{n}}{n!},
\]

\[
P = \left( \frac{1}{2} \right) (\Delta F) \cdot |J|^{2},
\]

where \( |J|^{2} \) is the energy density in the speckle.

b. We consider now the transparency of a scattering medium comprising a right parallelogram with dimensions \( L_{x} \times L_{y} \times L_{z} \). We assume the faces parallel to the x axis to be strongly reflecting. The light wave is incident on the medium in the x direction. The transparency (I) averaged over the realizations of the random potential (i.e., over the set of samples) is, in the zeroth approximation in the parameter \( \lambda / \xi \), given by

\[
I = \int \frac{e^{-n(r)}}{n!} dr.
\]

The integration in (9) is over the volume \( v = L_{x} \times L_{y} \times L_{z} \) of the parallelepiped. This yields an equation for the transparency fluctuations

\[
\langle (\delta I)^{2} \rangle = \left( \frac{1}{L_{x}^{2}} \right) \int \left( \frac{\delta n(r)}{n(r)} \right)^{2} \delta n(r) dr.
\]

Recognizing that

\[
\langle n \rangle = \frac{L_{x} L_{y} L_{z}}{\lambda},
\]

we obtain from (3)

\[
\left( \frac{\delta n(r)}{n(r)} \right)^{2} = \frac{1}{L_{x}^{2}} \int \delta n(r) dr.
\]

Substituting (1) in (10) we arrive at an expression for the fluctuations of a medium that is totally transparent

\[
\langle (\delta I)^{2} \rangle = \frac{2\pi L_{y} L_{z}}{L_{x}} I_{0}^{2}.
\]

Using (6), we can show that in the principal order in the parameter \( \lambda / \xi \) the distribution of the total optical transparency of a mesoscopic sample is Gaussian as long as \( \delta \theta / \theta \) is

If the wave incident on the sample is of duration \( T < R / \Delta \), then

\[
\left( \frac{\delta I}{I_{0}} \right)^{2} = \frac{2\pi L_{y} L_{z}}{L_{x}} I_{0}^{2}.
\]

Here \( I_{0} \) is the time-averaged sample transparency.

The interpretation of the foregoing results is similar to that in Refs. 10 and 11. The transparency of a sample under stationary illumination is determined by energy levels located in an energy band of order \( \hbar / \tau \) near the frequency of the incident light. Their number in the sample is of the order of \( N \sim v R / \tau \), where \( v \) is the density of states. The contribution of each level to \( I \) fluctuates from realization to realization by roughly a factor of two. In this case

\[
\langle (\delta I)^{2} \rangle = N \tau
\]

in accord with (12). If \( \tau < R / \Delta \), the sample transparency is determined by the quantum levels in an energy band \( \hbar / \tau \) and the energy sublevels of width \( \Delta E \) make independent contributions to \( \delta I \), so that

\[
\langle (\delta I)^{2} \rangle = N \tau
\]

in accord with (13).

c. The fluctuations considered above can be observed, in analogy with Refs. 12 and 13, by varying the incidence angle \( \theta \) of the light on the sample. A plot of \( \delta I / I(\theta) \) where \( \delta = \theta - \theta' \) is shown in Fig. 4. The correlator

\[
\langle J_{\alpha}(r \theta) J_{\beta}(r \theta') \rangle
\]

needed in this case is given by the diagrams 2a. The Green’s functions for the outer and inner loops correspond to the incidence angles \( \theta \) and \( \theta' \), respectively. At \( \delta = \theta - \theta' \) the value of \( \langle J_{\alpha}(r \theta) J_{\beta}(r \theta') \rangle \) is determined by the diagram 2c, which is the analog of the diagrams that give the conductance fluctuations of mesoscopic samples.6

A dependence similar to that plotted in Fig. 4 is obtained if \( I(\theta) \) is taken to mean the intensity, per unit incidence-light intensity, of the light scattered into an angle \( \theta \), given the incidence angle.

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Finally, the diffusion of the scatterers in the sample leads to random oscillations of $I$ with time. It is shown in Refs. 14 and 15 that if a typical scatterer is displaced a distance of order $A$ in a time of order $t_r$, a complete change of the realization takes place within a time $t_2 = (t_r/A)^2$, meaning that the speckle has an anomalous sensitivity to changes in the scattering positions.

To conclude this section we emphasize that the discussed fluctuations of the optical transparency differ in character from the mesoscopic conductance fluctuations considered in Refs. 5 and 6. In the latter was obtained at $T = 0$ the result

$$\langle (G/G_0) \rangle = e^4/\pi^2,$$

i.e.

It is seen from (14) that the relative transparency fluctuations are much larger than the relative conductance fluctuations. This result could be deduced from the Landauer formula\textsuperscript{16} that relates $G$ with the transparency $I(\theta)$ averaged over the angles $\theta$ of electron incidence on the surface of a disordered medium. As shown in Sec. IIc, the correlator $\langle J_{\text{ex}}(r,\theta)J_{\text{ex}}(r',\theta') \rangle$ is a rapidly decreasing function of $\delta = \theta - \theta'$. As a result, the contributions from diagrams 2a and 2b (i.e., of the extraneous currents proportional to $\langle n(r) \rangle$) to $\langle (G/G_0)^2 \rangle$ turn out to be small compared with whose of diagrams 2c in terms of the parameter $\lambda / L_0$. The contributions of these diagrams are independent of $\theta$ and are, roughly speaking, proportional to $\langle \nabla n(\theta) \rangle$. It is just the diagrams 2c which are the analogs of the diagrams summed in study (Refs. 5 and 6) of mesoscopic fluctuations of a linear response to an external electric field.

III. FLUCTUATIONS OF THE RESISTANCE OF MESOSCOPIC SAMPLES

a. The sizes of the current-density fluctuations produced in disordered metallic samples by an external electric field can also be obtained from Eqs. (1) and (2). In this case $n(r)$ and $J(r)$ have the meaning of the electron density and of the current density, while the correlator of the extraneous current is determined by averaging the microscopic currents over spatial scales of order $l$:

$$\langle J_{\text{ex}}(r)J_{\text{ex}}(r') \rangle = K_r + K_{r'} + K_n.$$

We have resolved the extraneous-current correlator into three terms corresponding to different physical mechanisms and represented by diagrams a, b, and c of Fig. 5:

\[ K_r = \frac{(4\pi D)^{1/2}}{(2\pi L_0)^{1/2}} \lambda \delta \langle \lambda (r-r') \rangle. \]

Here $D_n = (D/\lambda r_n)^{1/2}$ is the coherence length of the normal metal, and $L_n = (D/\lambda r_n)^{1/2}$ is the electron phase-loss length. Attention is called to the gigantic values of these currents, which are proportional to $L_0$. The physical cause of these fluctuations is that the response of the fluctuational current at a point $r$ to an electric field at a point $r'$ has at $T = 0$ a long-range character:\textsuperscript{17}

\[ I_n(r) = \int d^3x_n J_n(r',E_n(r')) \delta(r-r'), \]

whereas

\[ \langle \alpha_n(r, r') \rangle = \frac{1}{4\pi} \frac{\alpha_n(r-r')}{|r-r'|} \exp \left\{ \frac{|r-r'|}{l} \right\}. \]
We emphasize that in this case the fluctuational currents turn out to be much larger than the average current and the ergodicity problem was discussed in Ref. 17 with constant $E$, if the conductivity and

$$
\sigma_{\mu}(r) = \text{nonlocal conductivity, defined by the equation}
$$

$$
J(x) = \int \sigma_{\mu}(x, r') E(x, r') dr'.
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tion between the values of $\Delta \eta_1^{(11)}$ and $\Delta \eta_2^{(12)}$, measured with point contacts,

$$
\langle \Delta \eta_1^{(11)} \rangle = \langle \Delta \eta_2^{(12)} \rangle = \langle \Delta \eta \rangle L / L_c, \quad L < L_c.
$$

We examine now the expressions for $K_1$ and $K_2$ in (15):

$$
K_1 = - (4e^2 / h) \int E(r) E(r') f(r, r'), \quad (28)
$$

$$
K_2 = - (4e^2 / h) \int E(r) E(r') B(r, r'), \quad (29)
$$

$$
B(r, r') = \frac{dE}{dx} \frac{dE}{dx'} f(x) f(x') \left[ \delta_{L', L} \right], \quad (29a)
$$

which are obtained by summing the diagrams shown in Figs. 5b and 5c.

The physical essence of these terms is the long-range character of the correlations of the extraneous currents (in contrast to the $\delta$-correlated currents (16)). The amplitudes of these currents, however, are much smaller than in (16). As a result, the contributions of (28) and (29) to the conductance fluctuations $\langle \delta G \rangle^2$ are the same as that of (16).

These diagrams play no significant role in the calculation of the fluctuations of the electron density or of the scalar electric potential in small volumes near a point contact of size $a \ll L_c$, i.e., in the calculation of $\langle \delta \psi \rangle^2$.

The term (29) that corresponds to the diagrams c of Fig. 5 describes the long-range correlation of local (on a scale of order $L$) conductivities at the points $r$ and $r'$. Such a phenomenon can be studied by measuring the correlation of the intrinsic conductivities of the contacts $C_1$ and $C_2$:

$$
\langle G_1 G_2 \rangle = \frac{e^2}{h} \rho^2, \quad L < L_c, L_c.
$$

where $G_1$ and $G_2$ are respectively the conductances of the point contacts $C_1$ and $C_2$. In accordance with Refs. 5 and 6 we have

$$
\langle \delta G_1^2 \rangle = \langle \delta G_1 \rangle^2 = 0.
$$

The result obtained in Ref. 19

$$
\langle \delta G_1 \delta G_2 \rangle = \langle \delta G_1 \rangle \langle \delta G_2 \rangle
$$

is obtained also from the foregoing analysis, since $\langle \Delta \eta \rangle^2$ is independent of the location of the contacts.

e. Application of a magnetic field $H$ alters the character of the interference between the electron wave functions and leads as a result to fluctuations of $G(H)$ as a function of $H$. According to an ergodic hypothesis, $\langle \delta G \rangle^2$ is proportional to the density of states for a random potential of the same order. Experiment revealed in certain cases conductance fluctuations that exceeded $e^2 / h$ as $H$ varied. This can possibly be explained by the result (26) above.

Note that the conductances $G_I$ and $G_2$ of the point contacts $C_1$ and $C_2$ undergo fluctuations when the magnetic field is changed by an amount $\Delta H$, and $G_I$ and $G_2$ become destroyed by weak magnetic fields that change the values of $G_I$ and $G_2$ only little.

These phenomena can apparently be investigated by experiments on MIS structures, with the averaging over the realizations $\langle \ldots \rangle$ of the same order of $H$ as varied.

We discuss now the application of the Onsager relation to the conductance of mesoscopic samples in an external magnetic field. The standard deviation of the resistances $\delta R$ yields the quantity $\delta R$ as a function of $\Delta H$ at $H = 0$. Experiment revealed in certain cases conductance fluctuations that exceeded $e^2 / h$ as $H$ varied. This can possibly be explained by the result (26) above.

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the light energy at the given point of the medium. As a result, the transparency of mesoscopic samples fluctuates from sample to sample.

These fluctuations can be investigated by varying the angle of incidence of the radiation on the sample, in analogy to scattering from the sample and measuring the time variation of the transparency. The transparency is here anomalously sensitive to variation of the scatterer locations.\(^{14,15}\)

Another experimental possibility is to study the transparency fluctuations produced in samples by changes of the external magnetic field. The waves traveling along different paths lead to transparency fluctuations of the order of \(\text{order of magnitude of the transparency fluctuations produced in samples by changes of the}
\)

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\(^{1}\)The idea of such an interpretation belongs to B. I. Shklovskii.


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