Topological invariants in magnetohydrodynamics

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A formal definition of the process of reconnection of magnetic lines of force is given within the framework of almost ideal magnetohydrodynamics ($\mathrm{Re}_m \gg 1$). It refined previously published ideas. It is shown on its basis that the asymptotic Hopf invariant is conserved over time periods τ much shorter than the skin (diffusion) time τ_d. There are then in the general case no other invariants characterizing the magnetic field configuration in simply connected regions. A method of constructing the field-line-linkage invariants that are different from the Hopf invariant is given for the case of smooth flows of a perfectly conducting fluid ($\mathrm{Re}_m = \infty$).

INTRODUCTION

The freezing-in theorem[1] holds true in the course of the motion of a plasma with infinite conductivity, i.e., in the case when $\mathbf{E} = -[\mathbf{vH}]/c$. (Here and below, square brackets indicate vector products.) In particular, it follows from it that the topology of the configurations of the magnetic lines of force are preserved, i.e., the magnetic field lines can undergo only smooth deformation, with, in particular, no changes occurring in the linkage coefficients. Here it is, of course, assumed that the **v**, **H**, and **E** vector fields are sufficiently smooth.

When allowance is made for the finite conductivity within the framework of (isotropic) magnetohydrodynamics (MHD), the equation for **H** assumes the form

$$\partial\mathbf{H}/\partial t = \mathrm{curl}\,[\mathbf{vH}] - \mathrm{curl}\,(\nu\,\mathrm{curl}\,\mathbf{H}). \tag{1}$$

Here ν is the magnetic viscosity and **v** is the velocity of the medium. Generally speaking, Eq. (1) shows that in this case there do not exist any invariants characterizing the structure of the set of field lines. Indeed, for $\mathbf{v} = 0$, the magnetic field dissipates at $t = \infty$. In this case there remains after a sufficiently long period of time only the most weakly damped mode, $\mathbf{H}(\mathbf{x})$, the topological structure of whose lines of force does not depend on the topology of the field lines at $t = 0$.

Therefore, on the face of it, the subject of the present article seems to be trivial: for $\nu = 0$ any topological invariant is conserved, and for $\nu \neq 0$ no topological invariants exist. But the question is not that simple if we are interested in the rate of destruction of the invariants in the region of very small viscosities: $\nu \neq 0$.

Let us consider a plasma-magnetic field configuration with characteristic dimension L, whose evolution is described by Eq. (1) and the other implied MHD equations determining, in particular, the velocity field[1] $\mathbf{v}(\mathbf{x},t)$. In such a situation there are two characteristic time scales: the diffusion time $\tau_d = L^2\nu^{-1}$ and the hydrodynamic time $\tau_H = L/c_A$, where $c_A{}^2 = (H^2/4\pi\rho)^{1/2}$ is the Alfvén velocity, ρ being the plasma density. At small ν values the magnetic Reynolds number $\mathrm{Re}_m = \tau_d/\tau_H \gg 1$ is high. In this case the topological structure of the magnetic field can vary in the time period $t \ll \tau_d$. Indeed, it is possible that the nature of the MHD motions is such that there occurs a spontaneous sharp decrease in the spatial scales of the variation of **v** and **H** at times $t \sim \tau_H$. Similar examples are encountered in various problems: in turbulence, during the development of shock waves[2] and current sheets.[3] Therefore, the role of the second term in (1) is important at times $\tau \leqslant \tau_H \ll \tau_d$. In this case the topological structure of the field lines is not completely destroyed, as happens when $\tau \sim \tau_d$. Accordingly, the question of the existence of topological magnetic-field invariants that are conserved in time periods $\tau \ll \tau_d$ is entirely well-posed and quite important.

Taylor[4] was the first person to approach this question with similar formulations. His result can be formulated as follows: If there occurs in a system very small-scaled MHD turbulence that leads to a situation in which $\langle \mathbf{j}^2\rangle \gg \langle \mathbf{j}\rangle^2$ (here $\langle \ldots \rangle$ denotes spatial averaging), and if rapid topological changes are also possible in the magnetic-field configurations, then the following invariant is conserved:

$$h = \int_\Omega \mathbf{A}\cdot\mathbf{H}\,dV. \tag{2}$$

Here **A** is the vector potential of the field **H** ($\mathbf{H} = \mathrm{curl}\,\mathbf{A}$) and Ω is the total volume of the system, at the boundary of which $H_\perp = 0$. This invariant admits of an interpretation in terms of the linkage of the lines of force.[5,6] Following Arnol'd, we shall call it the asymptotic Hopf invariant. For simply connected regions, which are the only regions that will be considered in the present paper, relation (2) is gauge invariant. For multiply connected regions this is not so, which calls for minor technical complications [including limitations on the form of $\mathbf{A}(\mathbf{x})$]. All the results, with minor reformulations, remain valid in the general case.

Taylor's result can be reformulated. The rapid reconstruction of the topology of the field lines at $\tau \gg \tau_d$ can change the linkage of the individual field lines as a result of the small-scale turbulence, but preserves the overall linkage. Kadomtsev[7] formulated this result for magnetic-field configurations that vary during large-scale reconnections of the field lines.

In our paper we give a formal definition of the reconnection process in terms of cuts and splices. It allows us to impart precise meaning to the principal assertion concerning the conservation of the sole topological invariant—the asymptotic Hopf invariant—in the problem of rapid reconnection in simply connected regions. In multiply connected regions (tokamaks) the total magnetic fluxes are also conserved. Our definition gives specific form to Kadomtsev's ideas, eliminating the vagueness in the introduction of the reconnection concept.

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The paper is organized as follows. In Sec. 1, using specific examples, we consider the physical arguments leading to the concept of rapid reconnection of field lines under conditions when $\mathrm{Re}_m \gg 1$. In Sec. 2 we give new topological conservation laws for the configurations of magnetic field lines frozen into an ideal magnetic fluid, i.e., one with $\nu = 0$. In Sec. 3 we give the general definition of reconnection at low, but finite, values of ν, and show that only the asymptotic Hopf invariant is conserved.

1. PHYSICAL ARGUMENTS FOR THE INTRODUCTION OF THE RECONNECTION CONCEPT

In this section we shall consider the particular case of reconnection in the problem of current-sheet formation. It is the formalization of the corresponding properties that leads to the general definition of the reconnection concept for $\nu \to 0$.

In normal, almost ideal hydrodynamics discontinuities—shock waves—develop spontaneously in sufficiently fast flows. If the viscosity and thermal conductivity are sufficiently effective, the discontinuities are smeared out into fairly smooth structures. There are weighty arguments in support of the fact that in MHD, even in the highly nonideal, but $\nu = 0$ case, a specific mechanism allows for the appearance of surface-current carrying contact discontinuities—current sheets. This conclusion suggests itself when we analyze the equilibrium that develops from a given, initially nonequilibrium plasma-magnetic field configuration. This question has been considered in its most general formulation by Arnol'd.[6] He arrived at the conclusion that, among the smooth MHD equilibriums, i.e., those for which $[\mathrm{curl}\,\mathbf{H}, \mathbf{H}] + \Delta p = 0$, only a very narrow class (i.e., those governed by the equations $\mathrm{curl}\,\mathbf{H} = \alpha \mathbf{H}$, $\nabla \alpha = \nabla p = 0$) allow a nontrivial topological construction of a set of field lines. This class is so narrow that it is unlikely to contain equilibrium configurations for arbitrary initial complicated plasma-magnetic field configurations. Therefore, a general equilibrium configuration should contain discontinuities.

A similar question has been considered before from the dynamical standpoint by Syrovatskiĭ,[3] but the analysis is for a narrower class of two-dimensional problems with $p = \mathrm{const}$. If the initial magnetic configuration contains zero points, and goes over into a smooth equilibrium, then an arbitrarily weak perturbation will, generally speaking, prevent the initial configuration from going over into a state of smooth equilibrium. But the states containing two-dimensional, surface-current carrying strips—current sheets—will be equilibrium states (in the case when $\nu \equiv 0$). They are transverse to the initial plane.

It can be assumed that, in the general case of MHD with $\nu \equiv 0$, the plasma will, after going over into the equilibrium state, contain current sheets. If ν is arbitrarily small, but finite, then the second term in Eq. (1) cannot be neglected in the vicinity of the resulting current sheets. Therefore, magnetohydrodynamic equilibrium with $\nu = 0$ will not remain as such for $\nu \neq 0$.

The question of the nature of the resulting flow has not been fully clarified in the literature. The accepted point of view consists in the following. A flow develops with velocity v much higher than the diffusion rate c_A/Re_m, and this leads to fairly rapid reconnection of the magnetic fluxes lying on different sides of the current sheet. Such a flow has been considered by Petschek[8] (see Fig. 1). It contains four retarded shock waves, which intersect in the so-called diffusion region. The size of the diffusion region has not been unambiguously determined, but for the smallest admissible size the reconnection rate satisfies

$$v \sim c_A / \ln^\alpha \mathrm{Re}_m, \quad \alpha > 0.$$

As applied to the theory of solar flares,[9] the reconnection process has been considered in a diffusion region occupying the entire width of the current sheet; in this case we have $v \sim c_A \mathrm{Re}_m^{-1/2}$. The flow pattern differs only quantitatively from Fig. 1. For $\nu \to 0$ there also exist[10,11] exact MHD solutions describing the reconnection process; in this case $v \sim c_A$ holds. These solutions contain eight discontinuities that intersect at one point in the plane.

To avoid any misunderstanding, let us emphasize that, in the preceding paragraph, we discussed two-dimensional problems. The flows considered possess the following general properties.

The freezing-in conditions are well fulfilled in a small neighborhood of the point 0 (Fig. 1). The pairs of oppositely directed field lines move to this point, and, upon crossing this point, they "snap" and join in a different way. In the vicinity of a "splice" the field lines behave like the contour lines of a function in the vicinity of a saddle point. The characteristic reconnection time is $\tau_n = L/v \ll \tau_d$, and $\tau_n/\tau_d \to 0$ as $\mathrm{Re}_m \to \infty$. From this it follows that, on the scale of τ_d, during which the topology of the magnetic field relaxes completely, the reconnection process occurs instantaneously.

Kadomtsev[7] has noted that: a) qualitatively, the longitudinal magnetic field lying in the plane of the current sheet has no effect on the reconnection process; b) in the three-dimensional case the reconnection can occur inhomogeneously along the field line corresponding to the zero point of the two-dimensional problem.

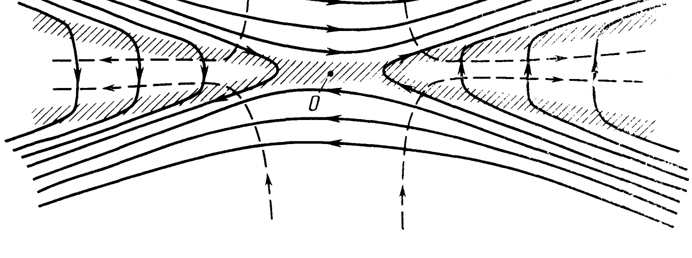


FIG. 1. Fast field-line reconnection pattern in the Petschek model. The continuous curves are magnetic field lines. The dashed curves are plasma streamlines depicting the flow of the plasma. The retarded shock waves and the diffusion region are hatched. Outside this region the magnetic field can be considered to be frozen into the plasma.

The above examples allow us to give a general definition of the reconnection process for $\nu \to 0$. The corresponding construction will be given in Sec. 3. First, let us consider the behavior of the field lines in ideal MHD with $\nu = 0$.

2. THE CONSERVATION LAWS IN A PERFECTLY CONDUCTING MAGNETIC FLUID

Equation (1) for ideal MHD is simpler:

$$\partial \mathbf{H}/\partial t = \text{curl}\,[\mathbf{v}, \mathbf{H}]. \tag{3}$$

It is easy to verify by direct computation that the integral invariant (2)—the asymptotic Hopf invariant—exists in this case:

$$h = \int_\Omega \mathbf{A}\cdot\text{curl}\,\mathbf{A}\,dV = \int_\Omega \mathbf{A}\cdot\mathbf{H}\,dV.$$

For an arbitrary **A** field the quantity h assumes a real value. If we consider the projection $f: S^3 \to S^2$, and identify the **A** field with the tangential field to S^3, then the invariant h is the classical Hopf invariant of the projection $f: S^3 \to S^2$, and h assumes integral values. The representation (2) for h in this case was obtained by Whitehead.[12]

The Hopf invariant admits of another topological interpretation, which was advanced by Hopf himself. Let there be prescribed the following regular mapping f (rank $f = 2$): $S^3 \to S^2$, and let x_0 and x_1 be regular points on S^2. The inverse images of the points x_0 and x_1 will respectively be the curves l_0 and l_1 in S^3. The Hopf invariant $h(f)$ of the mapping f is called the coefficient of linkage of the curves l_0 and l_1.

Arnol'd has shown[6] that the asymptotic Hopf invariant (on R^3) can be interpreted as the average (over all the field-line pairs) asymptotic number of linkages. He defined the latter in terms of the asymptotic behavior of the linkage of, possibly, open lines extended infinitely far, and "nonsingular" closure.

Let us note that, to prove the invariance of the expression (2), we do not need to carry out calculations. It is sufficient to bear in mind that, because of the freezing in of the field lines, the topology of the initial configuration (in particular, the linkage coefficients) is preserved at all times. But it is possible to construct examples of initial field-line configurations for which the linkage coefficients are equal to zero, although the configurations themselves remain connected (Fig. 2). An example of linkage for which the paired linkage coefficients are equal to zero, but the field lines cannot be unlinked, is the well-known—in lattice theory—Borromeo ring (Fig. 3). Following Ref. 13, we shall give the topological invariants of similar linkages, that can be regarded as

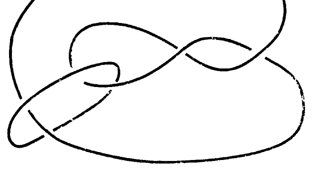


FIG. 2. The Whitehead linkage.

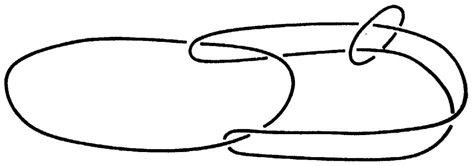


FIG. 3. The Borromeo ring.

higher conservation laws for the MHD equations in the case when h is equal to zero. In contrast to Ref. 13, we shall use tensor notation (which physicists are more familiar with), and not differential forms, although the latter are more suitable for topological investigations.

First, let us give explicit formulas for the solution of Eq. (3). These imply that the topological invariants of the linkages are conservation laws.

We shall describe the evolution of a magnetic field in a moving medium with the aid of the displacement field $\mathbf{X}(\mathbf{x},t)$ giving at the time t the position of that particle of the fluid which at the initial time $t = 0$ was located at the point $\mathbf{x}$. We thus introduce a Lagrangian description of the fluid motion. The displacement field generates at fixed t the one-to-one mapping

$$X:\ \mathbf{x} \to \mathbf{x}' = \mathbf{X}(\mathbf{x}, t).$$

The following relations are valid:

$$\begin{gathered} \mathbf{X}(\mathbf{x},0) = \mathbf{x}; \quad \frac{\partial}{\partial t}\mathbf{X}(\mathbf{x},t) = \mathbf{v}(\mathbf{X}(\mathbf{x},t),t); \\ \mathbf{X}^{-1}[\mathbf{X}(\mathbf{x},t),t] = \mathbf{x}; \quad \mathbf{X}[\mathbf{X}^{-1}(\mathbf{x},t),t] = \mathbf{x}. \end{gathered} \tag{4}$$

The introduction of such a mapping allows us to write down the solution to (3) in its explicit form[14]:

$$H_i(X,t) = H_j(\mathbf{x},0)\,\frac{\partial X_i}{\partial x_j}\,\det\left\|\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right\|. \tag{5}$$

In the case of a definite gauge, this change in the field H corresponds to the following change in the vector potential:

$$A_i(\mathbf{X},t) = \frac{\partial x_j}{\partial X_i} A_j(\mathbf{x},0). \tag{6}$$

The subscripts in (5) and (6) will hereinafter denote the coordinates of the vectors in some Cartesian coordinate system. In Eqs. (5) and (6) we have, for clarity, allowed some obvious simplification of notation.

Because of the reversibility of all the transformations entering into (5) and (6), all the topological invariants connected with $H(\mathbf{x},0)$ are, when $\mathbf{v}(\mathbf{x},t)$ is sufficiently smooth, the same as for $\mathbf{H}(\mathbf{x},t)$. Furthermore, it is easy to verify that, for the gauge employed, the X transformation leaves unchanged the following volume form: $(\mathbf{A}\cdot\mathbf{H})\,dV$, i.e., $\partial_t(\mathbf{A}\cdot\mathbf{H}) + \text{div}\{\mathbf{v}(\mathbf{A}\cdot\mathbf{H})\} = 0$, and $\mathbf{A}\cdot\mathbf{H}$ varies in the same way as the matter density.

Let us illustrate the construction of the topological invariants of linkages by carrying out the construction for the linkage of three curves: l_1, l_2, l_3. We shall need some topological facts (for proofs, see, for example, Ref. 15). For greater clarity, we shall begin with the consideration of the Hopf invariant for the linkage of the two curves l_1 and l_2; the invariant in this case coincides with the Gauss linkage coefficient.

Let $u_i^{(a)}$ be two covariant vector fields determined by the two curves l_a. Here $i = 1, 2, 3$ is the space index, while the index $a = 1, 2$ numbers the fields. Each $u_i^{(a)}$ field is defined on the complementary—to the curve l_a—set (which we shall denote by Ω_a) in S^3, and satisfies two characteristic conditions:

$$D_j u_i^{(a)} = 0 \quad (a=1,2); \qquad \int_C u_i^{(a)}\, dx^i = k(C, l_a).$$

Here and below $D_j u_{i_1,\ldots,i_n}$ denotes the antisymmetrized differentiation of the covariant tensors, i.e., $D_j u_{i_1,\ldots,i_n} = \partial_{[j} u_{i_1,\ldots,i_n]}$ (∂_j denotes the partial derivative $\partial/\partial x^j$) and $k(C,l_a)$ is the coefficient of linkage of a closed oriented curve C with l_a. The tensor $u_i^{(a)}$ is determined up to an additive gradient.

Let B_a ($a = 1, 2$) be the boundary of some tubular neighborhood of l_a, and let us assume that it does not intersect another curve. Then

$$\int_{B_1} u_i^1 u_j^2\, df^{ij} = -\int_{B_2} u_i^1 u_j^2\, df^{ij} = k(l_1, l_2). \tag{7}$$

Here df^{ij} is an element of surface area.

The expression for $k(l_1, l_2)$ can be represented in a form directly corresponding to (2):

$$\int_{S^3} u_i^1 v_{jk}\, dS^{ijk} = -\int_{S^3} v_{ij}^1 u_k^2\, dS^{ijk} = k(l_1, l_2), \tag{8}$$

where dS^{ijk} is an element of volume. In (8) v_{ij}^a is an antisymmetric covariant tensor, defined on $S^3 - l_a$, and

$$\int_Z v_{ij}^a\, df^{ij} = \operatorname{Ind}(Z, l_a). \tag{9}$$

Here Z is a two-dimensional oriented disk in S^3, and $\operatorname{Ind}(Z,l_a)$ is the index of the intersection of Z with l_a. Let us recall that the index of the intersection is defined as the algebraic sum of the number of intersections of the oriented curve with the disk. We omit the detailed topological definitions connected with the concepts introduced here. For details, see Ref. 13.

The coefficients $k(l_i, l_j)$ represent the numerical linkage invariants for $l = (l_1, \ldots, l_n)$. Let us define the first linkage coefficient for l as

$$\bar{k}(l) = \max_{ij} |k(l_i, l_j)|.$$

If l can be deformed into unlinked curves, then $\bar{k}(l) = 0$. But for a number of linkages, e.g., for the linkages shown in Fig. 3, $k(l) = 0$, but the linkages have not been undone. Let us introduce higher linkage coefficients. Consider the linkage $l = (l_1, l_2, l_3)$. From the condition $k(l_1, l_2) = 0$ it follows that there exist a covariant tensor u_i^{12} on Ω_{12} and antisymmetric covariant tensors v_{ij}^{12} and $v_{ij}'^{12}$ on ω_{12} (here ω^{12} is a compact set in Ω_{12}) such that

$$D_i \overset{12}{u}_k = \overset{1}{u}_{[i} \overset{2}{u}_{k]}, \qquad D_i \overset{12}{v}_{jk} = -\overset{1}{v}_{[ij} \overset{2}{u}_{k]}, \tag{10}$$
$$D_i \overset{\prime 12}{v}_{jk} = \overset{1}{u}_{[i} \overset{2}{v}_{jk]}.$$

Let $\bar{k}(l) = 0$ for $l = (l_1, l_2, l_3)$. It can be shown that the tensors

$$\tilde{u}_{jk}^{123} = \overset{12}{u}_{[j} \overset{3}{u}_{k]} + \overset{1}{u}_{[j} \overset{23}{u}_{k]},$$
$$\tilde{v}_{ijk}^{123} = -\overset{12}{v}_{[ij} \overset{3}{u}_{k]} + \overset{1}{v}_{[ij} \overset{23}{u}_{k]}, \tag{11}$$
$$\tilde{v}_{ijk}^{\prime 123} = \overset{12}{u}_{[i} \overset{3}{v}_{jk]} + \overset{1}{u}_{[i} \overset{23}{v}_{jk]}$$

satisfy the conditions

$$D_l \tilde{u}_{jk}^{123} = 0 = D_l \tilde{v}_{ijk}^{123} = D_l \tilde{v}_{ijk}^{\prime 123},$$

and that $\tilde{v}_{ijk}^{123}$ and $\tilde{v}_{ijk}^{\prime 123}$ are defined on the entire sphere S^3.

It is proved in Ref. 13 that there exists a whole number $k_2(l)$ such that

$$\int_{B_1} \tilde{u}_{ij}^{123}\, df^{ij} = -\int_{S^3} \tilde{v}_{ijk}^{123}\, dS^{ijk} = \int_{S^3} \tilde{v}_{ijk}^{\prime 123}\, dS^{ijk} = k_2(l). \tag{12}$$

It is this number that is called the second linkage coefficient. If these formulas are analyzed for the configurations of open curves, joined by, to use Arnol'd's terminology, "short" curves, then the formulas (12) give the asymptotic numbers of linkages of sets of three field lines. We can average them over all the sets of three field lines, and obtain new integral structure invariants for a magnetic field with $k(l_1, l_2) = 0$. This follows from the topological invariance of Eqs. (12) and the properties of $H(x,t)$, Eq. (5).

3. MAGNETIC FIELD LINE RECONNECTION IN SLIGHTLY NONIDEAL MHD

We shall define the reconnection process as the discrete transformation of the field $\mathbf{H}(\mathbf{x})$ at some fixed time $t = t_0$. We shall denote $\mathbf{H}_0(\mathbf{x},t_0 - 0)$ by $\mathbf{H}_0(\mathbf{x})$ and $\mathbf{H}_0(\mathbf{x},t_0 + 0)$ by $\mathbf{H}_1(\mathbf{x})$. Let M be that region in R^3 where a magnetic field $\mathbf{H}^0$ is prescribed so that it touches its boundary. Let us consider some "open" area π that is smooth and sufficiently small. Let a segment AB of some $\mathbf{H}_0$-field line lie in it (see Fig. 4). The line AB divides π into two parts: ACB and ADB. The area π comprises the "triangles" π_{ACB} and π_{ADB}, together with their common boundary. Let us denote by $\mathbf{n}$ the smooth field of unit normals to the surface of π. We choose π to be small, so that the conditions $(\mathbf{n},\mathbf{H}_0)|\pi_{ACB} > 0$, $(\mathbf{n}\cdot\mathbf{H}_0)\pi_{ADB} < 0$ and $(\mathbf{n},\mathbf{H}_0)|_{AB} = 0$ are fulfilled. Let

$$\pi_+ = \lim_{\varepsilon \downarrow 0} (\pi + \varepsilon n) \quad \text{и} \quad \pi_- = \lim_{\varepsilon \uparrow 0} (\pi + \varepsilon \mathbf{n})$$

be the two sides of the area π. We choose in M a sufficiently small subregion Ω containing π.

That discontinuous transformation $M\tilde{X}: \mathbf{x} \to \mathbf{x}' = \tilde{\mathbf{X}}(\mathbf{x})$, together with the corresponding transformation of $H(x)$, for which the following conditions are fulfilled is called a formal magnetic field line reconnection:

a) $\tilde{X}$ is smooth and reversible on the complement to $\bar{\pi}$ (where $\bar{\pi}$ is π, together with its boundary), and continuous and reversible at the boundary of π;

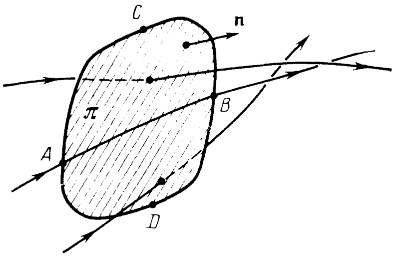


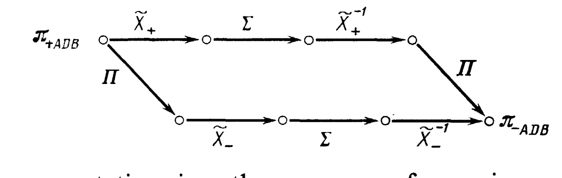
FIG. 4. The principal elements participating in the definition of reconnection. The area π is hatched. That side of this surface which faces us is designated in the text as π_+; the invisible side, as π_-. The vector $\mathbf{n}$ and three magnetic field lines are shown.

b) $\widetilde{X}$ is discontinuous on π;

c) $\widetilde{X}$ can be smoothly continued on π_+ and π_-. These continuations are denoted by $\widetilde{X}_+$ and $\widetilde{X}_-$;

d) Let σ_+ be the sum of the two areas $\widetilde{X}_+(\pi_{+ABD})$ and $\widetilde{X}_-(\pi_{-ADB})$; $\sigma-$, the sum of $\widetilde{X}_+(\pi_{+ACB})$ and $\widetilde{X}_-(\pi_{-ACB})$. Further, let σ_+ and σ_-, as surfaces in R^3, coincide. Denote this surface by σ. The relation between σ and $\sigma_\pm$ is the same as between π and $\pi_\pm$;

e) Let π be the canonical mapping $\pi_+ \to \pi_-$ and Σ be the canonical mapping $\sigma_+ \to \sigma_-$. It is assumed that the diagram



is commutative, i.e., the sequence of mappings $\pi_{+ADB} \to \pi_{-ADB}$ does not depend on the choice of the path;

f) the mapping $Q_+ : \pi_{+ACB} \to \pi_{+ADB}$, where $Q_+ = (\widetilde{X}_+{}^{-1}\Sigma\widetilde{X}_+)|\pi_{+ACB}$, preserves AB, and is a diffeomorphism;

g) outside σ the field $\mathbf{H}_1(\mathbf{x})$ is given in terms of $\widetilde{\mathbf{X}}(\mathbf{x})$ and $\mathbf{H}_0(\mathbf{x})$ by the formula (5).

It is assumed that the field $\mathbf{H}_1(\mathbf{x})$, which is initially defined only on the complement to σ, can be sufficiently smoothly continued to the entire M. (This, in particular, implies that on σ the normal component of $\mathbf{H}_1(\mathbf{x})$ is continuous and the total $\mathbf{H}_0$ flux through π equals zero.) This definition is the exact formulation of the intuitive reconnection picture described in Sec. 1.

Practically all the conditions in the definition are geometrically obvious. Let us, nevertheless, note the following. The conditions d) and e) imply, among other things, the reversibility of the above-introduced **H**-field transformation. The freezing-in condition obtaining outside the surface π (and, accordingly, σ) is formulated in paragraph g). Condition e) implies that, if two field lines l_1 and l_2 on different sides of AB each "snaps" into two parts, $l_1 \to l_1{}^+ l_1{}^-$ and $l_2 \to l_2{}^+ + l_2{}^-$, and the two halves $l_1{}^+$ and $l_2{}^-$ are "spliced," then the remaining field-line "tails" $l_1{}^-$ and $l_2{}^+$ also get spliced (see Fig. 4). The condition f) implies that the reconnection process can be represented as a composition of a large number of infinitely small reconnections.

From this definition follows the principal result of the paper, namely, the conservation of only one topological invariant—the asymptotic Hopf invariant. This result can be obtained with the aid of the correspondence, introduced by Arnol'd,[6] between the asymptotic Hopf invariant and the asymptotic number of linkages of pairs of field lines. But the analytical proof presented below is technically simpler.

Let us denote the complement in M of the area σ figuring in the definition of reconstruction by M_σ. Consider any smooth vector potential $\mathbf{A}_0(\mathbf{x})$ of the field $\mathbf{H}_0(\mathbf{x})$. We introduce in M_σ the field $\mathbf{A}_1(\mathbf{x})$, which can be computed from $\mathbf{A}_0(\mathbf{x})$ and $\widetilde{X}(\mathbf{x})$ in accordance with (6); we shall denote its smooth continuation into $\sigma_\pm$ in the same way. In M_σ, $\mathbf{H}_1(\mathbf{x}) = \text{curl}\,\mathbf{A}_1(\mathbf{x})$. Since **A** and **H** vary like density in a fluid element,

$$\int_{M_\sigma} \mathbf{A}_1 \cdot \mathbf{H}_1\, dV = \int_M \mathbf{A}_0 \cdot \mathbf{H}_0\, dV. \tag{13}$$

If $\mathbf{A}_1(\mathbf{x})$ can be considered to be the vector potential of the field $\mathbf{H}_1(\mathbf{x})$ in the entire M, then the identity (13) proves the invariance of the asymptotic Hopf invariant. But for arbitrary $\mathbf{A}_0(\mathbf{x})$ the tangential—to σ—$\mathbf{A}_1(\mathbf{x})$ component, which we shall denote by $\mathbf{A}_{1t}$, turns out to be discontinuous in σ, and $\text{curl}\,\mathbf{A}_1(\mathbf{x}) \neq \mathbf{H}_1(\mathbf{x})$ in σ, or, more exactly, $\text{curl}\,\mathbf{A}_1$ contains, generally speaking, δ-function discontinuities in σ, which, in particular, leads to an indeterminacy in the expression

$$\int_M \mathbf{A}_1 \cdot \text{curl}\,\mathbf{A}_1\, dV.$$

It nevertheless turns out that we can, by using condition e), choose $\mathbf{A}_0$ so that $\mathbf{A}_{1t}$ will be continuous and sufficiently smooth in σ. In order to show this, let us introduce in π_{ACB} a smooth function φ such that: a) its smooth continuation into both AB and ACB is strictly monotonic; b) $|\nabla_t \varphi| \neq 0$. Here ∇_t is the gradient on the π_- surface. Let us continue φ into π_{ADB} in the following manner:

$$\varphi(P)|_{\pi_{ADB}} = \varphi(\widetilde{X}^{-1}\Sigma^{-1}\widetilde{X}(P)).$$

In accordance with condition e), the continuation of φ into π_{ADB} does not depend on which side of π_{ADB} we take the point P. In accordance with this definition of φ, segments of any contour of φ that lie on different sides of AB "get spliced" in σ after the transformation $\widetilde{X}$. Let us now choose $\mathbf{A}_0$ so that $\mathbf{A}_{0t} \| \nabla_t \varphi$. If this condition is not fulfilled for $\mathbf{A}_0'$, then we can choose $\psi(\mathbf{x})$ so that $\mathbf{A}_0 = \mathbf{A}_0' + \nabla\psi$ satisfies it.[1)] This condition imposes limitations only on the values of φ in π. For a curve γ belonging to π_{+ADB} we have

$$\int_\gamma \mathbf{A}_0 d\mathbf{x} = \int_{\widetilde{X}^{-1}\Sigma^{-1}\widetilde{X}(\gamma)} \mathbf{A}_0 d\mathbf{x} = \int_{\widetilde{X}(\gamma)} \mathbf{A}_1 d\mathbf{x} = \int_{\Sigma^{-1}\widetilde{X}(\gamma)} \mathbf{A}_1 d\mathbf{x}. \tag{14}$$

Here, besides the properties of $\mathbf{A}_0(\mathbf{x})$, we have used: a) the nondependence of $\int \mathbf{A} d\mathbf{x}$ on the time if the curve (which may be open) moves together with the fluid and $\mathbf{A}(\mathbf{x})$ satisfies (6), and b) the continuity of the normal—to σ—component of $\mathbf{H}_1$. Thus, $\mathbf{A}_{1t}$ is continuous, and, accordingly, $\mathbf{H}_1 = \text{curl}\,\mathbf{A}_1$. As noted above,

$$h_1 = \int_M \mathbf{A}_1 \cdot \text{curl}\,\mathbf{A}_1\, dV = \int_M \mathbf{A}_1 \cdot \mathbf{H}_1\, dV = \int_{M_\sigma} \mathbf{A}_1 \cdot \mathbf{H}_1\, dV = \int_M \mathbf{A}_0 \cdot \mathbf{H}_0\, dV = h_0,$$

which is what was required to be proved.

Let us now consider the change $\mathbf{H}(\mathbf{x},0) \to \mathbf{H}(\mathbf{x},t)$ in the magnetic field, such that it generates a finite combination (composition) of diffeomorphisms of M and reconnections. The latter, according to the comment made above, can be extended in time. Thus, the resulting transformation of $\mathbf{X}(\mathbf{x},t)$ generates the transformation $\mathbf{H}(\mathbf{x},0) \to \mathbf{H}(\mathbf{x},t)$. Such changes in the field simulate, according to the arguments adduced in Sec. 1, the changes that are found to occur in H at times $\tau \ll \tau_d$ within the framework of MHD with $\nu \to 0$. It can be conditionally assumed that $\mathbf{H}(\mathbf{x},t)$ and $\mathbf{v}(\mathbf{x},t) = \partial \mathbf{X}[\mathbf{X}^{-1}(\mathbf{x},t), t]/\partial t$ are connected by the equation $\partial \mathbf{H}/\partial t = \text{curl}[\mathbf{vH}]$, which can now be understood in the generalized sense.

The above arguments show that the asymptotic Hopf invariant, which is related[6] to the asymptotic coefficients of linkage of field-line pairs, is conserved during such magnetic-field evolution. Of importance here is the existence of an integral expression for the mean number of linkages. In spite of the fact that the individual linkage coefficients change during the reconnection process, the quantity $k(l_a, l_b)$ averaged over all the (l_a, l_b) pairs, i.e., the asymptotic Hopf invariant, does not change.

The situation is somewhat different in the case of the higher linkage coefficients k_i (the formula for the second linkage coefficient is given in (12)). An integral representation of the type (12) obtains in the case when $\bar{k}$ is equal to zero. It is clear that the reconnection does not preserve the configurations with nontrivial coefficients k_i. Indeed, linked pairs of field lines generally speaking arise in the course of the reconnection, and this destroys the integral representations of the type (12).

Next, let us introduce some sufficiently narrow class of magnetic configurations, and show that, if we allow magnetic-field evolution accompanied by reconnection, then no topological invariants, except the asymptotic Hopf invariant (in simply connected regions), are conserved on this class. Let us consider a magnetic-field configuration with $h=0$, consisting of a finite number of closed tubes of force with identical fluxes and the following property: In their interior they differ from a set of unlinked field lines that close up on going around once along a tube only by rotation through a finite angle about the tube axis at some cross section of the tube, i.e., we can introduce at each cross section of a tube a "polar" coordinate system such that all the coordinate lines go over into coordinate lines in the transformation generated by traversing the circumference of the tube along field lines, i.e., the interior of each tube is constructed like a deformed "tokamak" with a constant rotational transform angle.

The configuration under consideration can be reduced to a single unknotted tube through a finite number of reconnections. Each reconnection gives rise to an additional rotation of the field lines inside the tubes through a constant angle. Therefore, all the field lines inside the obtained tube will be rotated through some constant angle relative to the tube "axis." The Hopf invariant is conserved in such a transformation, and the angle of rotation is proportional to it. Therefore, this angle is equal to zero, and all the field lines in the resulting tube will be unlinked. This argument shows that there are no other invariants, except the Hopf invariant, in the class of fields under consideration. It can be carried over to configurations with arbitrary $h \neq 0$.

The class of magnetic fields considered, including those with $h \neq 0$, apparently possesses the property that any field $\mathbf{H}$ can be approximated by fields from this class with an arbitrarily high degree of accuracy. Consequently, there are no invariants (except the asymptotic Hopf invariant) that continuously depend on $\mathbf{H}(\mathbf{x})$. From the physical standpoint, this implies the absence of any other invariants, except the asymptotic Hopf invariant, in processes that admit of reconnection in the sense defined by us. Let us recall that the individual total magnetic-field fluxes are also conserved in multiply connected regions.

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[1)]The possibility of such a choice of ψ is due to the fact that, for example, there exists in the (x,y) plane such a $\chi(x,y)$ for which $\partial\chi/\partial x$ assumes prescribed values.

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