Angular dependence of the critical field of superconducting superlattices: theory

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A theory is derived for the angular dependence of the upper critical field $H(\theta)$ of a superlattice formed by thin superconducting layers with Josephson junctions between them. The function $H(\theta)$ is singular at small angles at all temperatures. Expressions are derived for the quantity $\beta(H) = dH/d\theta|_{\theta=0}$, which characterizes the singularity. The functional dependence $H(\theta)$ is found for all angles in the region of the "two-dimensional behavior" of the superlattice (low temperatures). The change in $H(\theta)$ is found to be sharper than that predicted by the Tinkham formula for thin films. The effect of nonideal superlattice structure on $H(\theta)$ is studied.

INTRODUCTION

Research on semiconducting superlattices has recently been supplemented with active research on superlattices containing layers of superconducting material. A superstructure in a solid modifies the spectra of elementary excitations and gives rise to changes in several macroscopic properties.

The most characteristic feature of the static properties of superlattices is the unusual temperature dependence of the upper critical magnetic field, $H_{\parallel}(T)$, in the case in which the layers are oriented parallel to the field ($\theta = 0$). As the temperature is lowered near a certain value $\tilde{T}_c < T_c$ (T_c is the temperature of the superconducting transition in a zero field), a crossover occurs in the behavior of $H_{\parallel}(T)$: a transition from a linear dependence to a square-root dependence.[1–3] This transition is extremely sharp.[3] It is usually assumed[1–4] that the properties of a superlattice in the "three-dimensional region" ($T > \tilde{T}_c$) are the same as the properties of an anisotropic bulk superconductor, while those in the "two-dimensional region" ($T < \tilde{T}_c$) are the same as the properties of a single film. We know that the angular dependence of H_c for a thin film is quite different from that of an anisotropic bulk superconductor; the two cases differ fundamentally when the angular deviation θ from parallel orientation is small. For an unbounded bulk medium we would have $H(0) - H(\theta) \propto \theta^2$, while for a thin film $H(\theta)$ would be a singular function: $H(0) - H(\theta) \propto |\theta|$.

The behavior $H(\theta)$ observed experimentally[2,5,6] usually deviates both from the Tinkham thin-film formula[7] for the two-dimensional case and from the Lawrence-Doniach formula[8] for the three-dimensional region, in the direction of a sharper change in the critical field as a function of the angle.

In this paper we analyze theoretically the angular dependence of the critical field of a superlattice with dielectric interlayers, which provide Josephson junctions between the superconducting layers. We will see that the intuitive assumption that the Tinkham formula would be valid for a superlattice in the two-dimensional region is not correct, because of a peculiar proximity effect between adjacent layers of the superlattice. An expression derived with allowance for this effect does indeed predict a sharper change in the critical field as a function of the angle than that predicted by the Tinkham formula.

The discrete structure of the superlattice causes the $H(\theta)$ functional dependence to be singular as $\theta \to 0$ at all temperatures, including the region of three-dimensional behavior. We derive expressions for the quantity $\beta = dH/d|\theta||_{\theta=0}$, which characterizes this singularity, as a function of the magnetic field for the regions of both two-dimensional and three-dimensional behavior. We find the width of the characteristic range of angles in which the Lawrence-Doniach formula does not apply.

We will see that in a defective superlattice, with deviations from strict periodicity, the curves of $H(\theta)$ at small angles are sharper than those for an ideal superlattice. As the defect in the superlattice we consider a single metal layer in the volume of a superlattice with a thickness which differs from the thicknesses of all the other layers; we also consider the superlattice boundary.

To study the functional dependence $H(\theta)$ in the limit $\theta \to 0$ we use the method of adiabatic separation of variables in the linearized Ginzburg-Landau equation. This method results in a simple and graphic derivation of exact expressions for $\beta(H)$ in all the cases mentioned above.

This work was carried out in parallel with an experimental study of the angular dependence of the critical field for V/Si superlattices. The results of the experimental study are reported in the second part of this study.[9] In that other paper we also compare the experimental results with the theoretical results of the present paper.

1. ADIABATIC METHOD IN THE PROBLEM OF THE ANGULAR DEPENDENCE OF THE CRITICAL FIELD

A slab inhomogeneity in a superconductor influences the orientational dependence of the critical magnetic field H. When the angle θ between the field and the plane of the inhomogeneity is 0 or $\pi/2$, the problem of the form of a superconducting nucleating region can be solved relatively easily, thanks to the symmetry. For an arbitrary angle θ, the functional dependence $H(\theta)$ can be found at best approximately, for example, by variational methods.[10] The only exceptional case is that of orientations which deviate only slightly from parallel. We will develop a simple method for determining the limiting behavior of $H(\theta)$ exactly in the limit $\theta \to 0$, without reference to the nature of the inhomogeneity (which might be, in particular, the boundary of the sample or a regular arrangement of layers in the superlattice). The functional dependence $H(\theta)$ is found to be singular; as a rule, we have $H(0) - H(\theta) \propto |\theta|$.

Determining the critical field reduces to solving a lin-

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earized Ginzburg-Landau equation. Let us assume that the field H is oriented along x, while the plane of the inhomogeneity is rotated by an angle $\theta \ll 1$ with respect to H around the y axis. The Ginzburg-Landau equation then takes the form

$$-\xi^2\frac{\partial^2\psi}{\partial x^2}+\left\{-\xi^2\left(\frac{\partial}{\partial y}+\frac{2\pi i}{\Phi_0}Hz\right)^2-\xi^2\frac{\partial^2}{\partial z^2}+\hat{u}(z-\theta x)\right\}\psi=\tau\psi. \tag{1.1}$$

Here ξ is the coherence length at $T=0$; $\Phi_0 = \pi\hbar c/e$ is the quantum of flux; $\tau = (T_c - T)/T_c$, where T_c is the transition temperature at $H=0$; and the operator $\hat{u}$ characterizes the inhomogeneity.

Basing our approach on the analogy between the linearized Ginzburg-Landau equation (1.1) and the Schrödinger equation, we borrowed the idea of adiabatic separation of variables from quantum mechanics[11] in order to calculate $H(\theta)$. The dimensionless temperature of the superconducting transition in a magnetic field, τ, in (1.1) corresponds to the smallest energy eigenvalue of the motion of the particle determined by the Schrödinger equation. The part of Eq. (1.1) in braces corresponds to the energy of the motion transverse to the field H. The energy of the "longitudinal" motion becomes progressively smaller in comparison with the energy of the "transverse" motion as the angle θ decreases. It is this circumstance which makes it possible to use the adiabatic separation of variables. We first determined the eigenvalue $\tau_\perp$ and the eigenfunction $\psi_x(y,z)$ of the operator

$$\hat{\mathscr{H}}_0=-\xi^2\left(\frac{\partial}{\partial y}+\frac{2\pi i}{\Phi_0}Hz\right)^2-\xi^2\frac{\partial^2}{\partial z^2}+\hat{u}(z-\theta x), \tag{1.2}$$

treating the variable x as a fixed parameter.

If $\psi(y,z) = e^{iky}\varphi(z,k)$ and $\tau_\perp = \tau_\perp(H,k)$ are the eigenfunction and eigenvalue for $\theta = 0$, at nonzero θ we have

$$\psi_x(y,z)=e^{iky}\varphi(z-\theta x,\ k+2\pi\theta Hx/\Phi_0),$$
$$\tau_\perp=\tau_\perp(H,\ k+2\pi\theta Hx/\Phi_0). \tag{1.3}$$

An inhomogeneity affects the critical field at $H=0$ only if a nucleating region is localized near this inhomogeneity. Localization corresponds to the existence of a minimum on the curve of $\tau_\perp$ versus k at some $k = k_0(H)$. The most typical case would be a quadratic dependence $\tau_\perp(k)$ near $k = k_0(H)$. Assuming $k = k_0(H)$ in (1.3), we expand $\tau_\perp$ in a series in x:

$$\tau_\perp=\tau_\perp(H)+\frac{1}{2}\frac{\partial^2\tau_\perp}{\partial k^2}\left(\frac{2\pi\theta Hx}{\Phi_0}\right)^2+\dots \tag{1.4}$$

A substitution of the eigenvalue (1.4) for the operator $\mathscr{H}_0$ in (1.1) shows that the $\tau_\parallel$ spectrum, of the energies of the motion along the field, changes qualitatively when θ undergoes an arbitrarily small deviation from 0. Specifically, at $\theta = 0$ the spectrum is continuous, and the smallest value, $\tau_\parallel = 0$, corresponds to a spatially uniform solution, while for $\theta \neq 0$ the spectrum of the longitudinal motion becomes discrete, and the lowest level corresponds to the energy of the zero-point vibrations along the x axis. In this case we have $\tau_\parallel > 0$. The change in the nature of the spectrum makes the critical field a nonanalytic function of the angle θ. As we will see below, the amplitude of the zero-point vibrations is small enough that we can restrict the discussion to the quadratic expansion in (1.4). The equation determining $\tau_\parallel$ is thus the Schrödinger equation for a harmonic oscillator:

$$-\xi^2\frac{d^2\psi(x)}{dx^2}+\frac{1}{2}\frac{\partial^2\tau_\perp}{\partial k^2}\left(\frac{2\pi\theta Hx}{\Phi_0}\right)^2\psi(x)=\tau_\parallel\psi(x). \tag{1.5}$$

Using the smallest eigenvalue $\tau_\parallel$ obtained from (1.5), we can easily determine the functional dependence $\tau(H,\theta)$ at small values of θ, and we can determine the derivative

$$\beta=\left.\frac{\partial H}{\partial|\theta|}\right|_{\theta=0}=-\left(\frac{1}{2}\frac{\partial^2\tau_\perp}{\partial k^2}\right)^{1/2}\left(\frac{\partial\tau_\perp}{\partial H}\right)^{-1}\frac{\pi\xi H}{\Phi_0}. \tag{1.6}$$

The function $\psi(x)$ found from (1.5) decays over a characteristic distance $x_0 \propto \theta^{-1/2}$. Consequently, the quantity θx in which the expansion is carried out in (1.4) is small in the limit of small angles ($\theta x_0 \propto \theta^{1/2}$), so that we can restrict the discussion to the quadratic potential in (1.5).

The adiabatic eigenfunction $\psi(x,y,z) = \psi(x)\psi_x(y,z)$ which we have constructed is applicable if the component of the kinetic energy associated with the longitudinal motion, $-\xi^2 d^2/dx^2$, is substantially larger by virtue of the function $\psi(x)$ than the component which comes from the parametric x dependence of the function $\psi_x(y,z)$. This condition can be formulated as an inequality which states that the distance between the eigenvalues $\tau_\parallel$ is small in comparison with the scale of the variation, $\tau_\perp(k)$. For superlattices with superconducting layers of thickness d, the adiabatic approximation can be used under the condition

$$|\theta|\ll\frac{d^2}{\xi}\frac{H}{\Phi_0}\left[\frac{\partial^2\tau_\perp(H,k)}{\partial k^2}\right]^{1/2}. \tag{1.7}$$

It follows from (1.5) that H is a linear function of $|\theta|$ as $\theta \to 0$. In certain special cases, e.g., for a plate in a parallel field equal to the vortex entry field,[7] the quadratic term may be absent from expansion (1.4). The result will be a change in the functional dependence $H(\theta)$. It is easy to show that if the expansion (1.4) begins with the term of order $2n$ then we would have

$$H(0)-H(\theta)\propto|\theta|^{2n/(n+1)},\quad n=1,2,3,\dots \tag{1.8}$$

Expressions (1.6) and (1.8) are valid for an inhomogeneity of arbitrary type, in particular, for the boundary of a sample. The effect of a boundary on the angular dependence of the critical field has been studied in several theoretical papers.[12] The result derived by Saint-James *et al.*[7] agrees with (1.6) of the present paper in the limit $T \to T_c$, with $\partial^2\tau/2\partial k^2 \approx 1$. The reasons for the difference between the result of Ref. 7 and the exact result—a difference which increases with decreasing temperature—were pointed out by Thompson.[12] In that paper Thompson derived an expression equivalent to (1.6) through a rather complicated analysis of the perturbation-theory series for the case of a plate with ideal boundaries. The derivation which we have set forth here not only is clearer than the series analysis of Ref. 12 but also allows one to immediately see the behavior $H(\theta)$ in the particular case in which the field $H(0)$ is equal to the vortex entry field H_V: In this case we have $n = 2$ and $H(0) - H(\theta) \propto |\theta|^{4/3}$.

2. ANGULAR DEPENDENCE OF THE CRITICAL FIELD FOR AN IDEAL SUPERLATTICE

We examine the behavior of $H(\theta)$ for a strictly periodic superlattice obtained by alternating metal layers of thickness d and layers of an insulator (or semiconductor) of thickness s. The coupling between the metal layers is assumed to be weak, and this weakness is reflected in the boundary conditions where two successive periods of the superlattice adjoin each other:

$$\psi'(Dn+d/2)=\psi'[D(n+1)-d/2],$$
$$\psi[D(n+1)-d/2]-\psi(Dn+d/2)=l\psi'(Dn+d/2). \quad (2.1)$$

The extrapolation length l, which is inversely proportional to the energy of the Josephson junction,[13] is large: $l \gg d$. The period is $D = d + s$. If, in addition, the metal layers are too thin for vortices to penetrate into them [$d \ll L_H$, where the magnetic length is $L_H = (\Phi_0/2\pi H)^{1/2}$], there is no difficulty in deriving a finite-difference equation for the order parameter ψ_n in the layers.[3,14] If terms proportional to the product[14] of small parameters, $(d/l)(D/L_H)^2$, are ignored, the equation for $\psi_n(x,y)$ becomes

$$-\frac{\xi^2}{ld}(\psi_{n+1}+\psi_{n-1}-2\psi_n)-\xi^2\left\{\frac{\partial^2}{\partial x^2}+\left(\frac{\partial}{\partial y}+\frac{2\pi iHDn}{\Phi_0}\right)^2 -\frac{\pi^2}{3}\left(\frac{Hd}{\Phi_0}\right)^2\right\}\psi_n=\tau\psi_n. \quad (2.2)$$

(The superlattice layers run parallel to the xy plane; the magnetic field is oriented along the x axis.) It follows from (2.2) that the spectrum $\tau_\perp(H,k)$ is determined by the equation

$$-\frac{\xi^2}{ld}(\psi_{n+1}+\psi_{n-1}-2\psi_n) +\xi^2\left[\left(k+\frac{2\pi HDn}{\Phi_0}\right)^2+\frac{\pi^2}{3}\left(\frac{Hd}{\Phi_0}\right)^2\right]\psi_n=\tau_\perp\psi_n. \quad (2.3)$$

The minimum eigenvalue $\tau_\perp$ corresponds to $k = 0$. We know[3,14] that in the behavior $\tau_\perp(H)$ and thus in that of the parallel critical field $H_\parallel(T)$ there is a crossover corresponding to a transition from three-dimensional to two-dimensional behavior. For our purposes below, it is convenient to introduce a characteristic crossover field[14] H_{cr} in (2.3),

$$H_{cr}=\Phi_0/\pi D(dl)^{1/2}, \quad (2.4)$$

and to transform from k and $\tau_\perp$ to the dimensionless parameters r and $\lambda(H,k)$:

$$r=k\Phi_0/\pi HD, \quad \lambda(H,r)=(\Phi_0/2\pi D\xi H)^2\tau_\perp(H,k). \quad (2.5)$$

The quantity λ depends periodically on r. One period corresponds to values $|r| \leqslant 1/2$. In terms of dimensionless variables, Eqs. (2.3) and (1.6) take the form

$$-(H_{cr}/2H)^2(\psi_{n+1}+\psi_{n-1}-2\psi_n)+[(n-r)^2+d^2/12D^2]\psi_n=\lambda\psi_n, \quad (2.6)$$

$$\beta=-\left(\frac{1}{2}\frac{\partial^2\lambda}{\partial r^2}\right)^{1/2}\left(2\lambda+H\frac{\partial\lambda}{\partial H}\right)^{-1}\frac{\Phi_0}{2\pi D^2}. \quad (2.7)$$

The equation written in the form (2.6) is convenient for analyzing the low-temperature region, where H_{cr}/H is a small parameter. In this H region, it suffices in calculating $dH/d\theta$ to lowest order to consider Eq. (2.6) for ψ_0 only and to set $\psi_1 = \psi_{-1} = 0$. We then find $\beta = -3\Phi_0/\pi d^2$ from (2.7); this value agrees with the value for an isolated thin film.[7,12] To derive the first nonvanishing corrections to this result, we should use an iteration method to solve the system of equations for ψ_0 and $\psi_{\pm 1}$, ignoring $\psi_{\pm 2}$. The calculations yield

$$\beta(H)\approx-\frac{3\Phi_0}{\pi d^2}\left\{1-\frac{1}{4}\left(6\frac{D^2}{d^2}+1\right)\left(\frac{H_{cr}}{H}\right)^4\right\}. \quad (2.8)$$

Expression (2.8) shows that for a superlattice the correction to the "thin-film" value of β increases rapidly with increasing temperature (i.e., with increasing field H). In agreement with (1.7), the linear asymptote of $H(\theta)$ prevails at angles $|\theta| \ll \pi d^2 H/2\Phi_0$.

In analyzing the behavior $\beta(H)$ in weak field ($H \ll H_{cr}$), it is convenient to use Fourier transforms, putting (2.6) in the standard form[15] of the Mathieu equation.[4,16] In the strong-coupling approximation we find the following from the Mathieu equation:

$$\lambda(H,r)=\lambda(H)+\Delta\lambda(1+\cos 2\pi r)/2. \quad (2.9)$$

Using the zone width $\Delta\lambda$ found by the WKB method,[15] and using (2.9) to calculate the derivative $\partial^2\lambda/\partial r^2$, we find from (2.7)

$$\beta(H)=-\frac{2^{7/4}\Phi_0}{\pi^{1/4}D^2}\left(\frac{H}{H_{cr}}\right)^{1/4}\cdot\exp\left\{-2\frac{H_{cr}}{H}\right\}. \quad (2.10)$$

In weak fields the quantity $\beta(H)$ is thus an exponential function of the field. Figure 1 shows a curve of $\beta(H)$ over the entire field range for the case $D = d$.

For superlattices, the angular interval in which the linear behavior $H(|\theta|)$ is valid depends strongly on the temperature. For $H \ll H_{cr}$, the $H(\theta)$ dependence for the superlattice as a whole is similar to that for a homogeneous but isotropic superconductor. Near $\theta = 0$, this function is dome-shaped. In contrast, however, expression (2.10) reveals a singularity at $\theta = 0$. Actually, the expression for a homogeneous but anisotropic superconductor describes the behavior $H(\theta)$ for a superlattice with a certain deviation from a parallel orientation ($|\theta| \gtrsim \theta^*$). For an ideal superlattice it is simple to evaluate θ^*, since we can construct[16,17] a one-dimen-

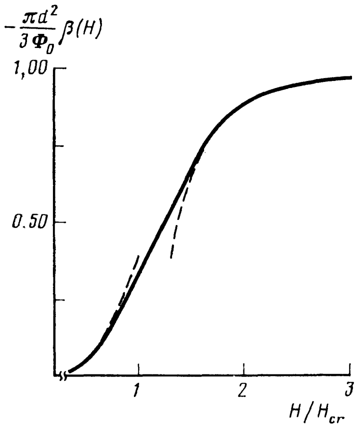

FIG. 1. The normalized derivative $(\pi d^2/3\Phi_0)(dH/d|\theta|)|_{\theta=0}$ versus the reduced magnetic field H/H_{cr} for the case $D = d$. The dashed lines show the asymptotic behavior of (2.8) and (2.10) in the region $H \approx H_{cr}$.

sional Ginzburg-Landau equation which determines $H(\theta)$ for any angles θ. In the coordinate system of the superlattice [the plane of the layers coincides with the xy plane; $H=(H_x,0,H_z)$] this equation becomes

$$\left\{-\frac{d^2}{dt^2}+\frac{1}{2}\left(\frac{H_{cr}}{H}\right)^2(1-\cos t)+\frac{d^2}{12D^2}\right\}\psi$$
$$+\left[\frac{\Phi_0 H_z}{2\pi D^2 H_x^2}\right]^2(t+t_0)^2\psi=\lambda\psi. \qquad (2.11)$$

Here λ is determined by (2.5) with the obvious substitution $H\to H_x, \tau_\perp(H,k)\to\tau_\perp(H,t_0)$; and the dimensionless coordinate is $t=(2\pi DH_x/\Phi_0)y$. The quantity λ depends periodically on t_0: $\lambda(t_0)=\lambda(t_0+2\pi)$. The terms in braces in (2.11) constitute an energy operator with a periodic potential, which forms a band spectrum. Since we are interested in the smallest eigenvalues, we set $t_0=0$ at this point. The second term on the left side of (2.11) distorts the periodic potential. If this is a smooth distortion, its effect on the spectrum of λ values can be treated in the effective-mass approximation,[18] which is equivalent to the adiabatic approximation used above. The adiabatic approximation is thus valid as long as the smooth quadratic increment in the periodic potential remains small in comparison with the band width $\Delta\lambda$ as t increases. Since the energy of the zero-point vibrations is determined by the region of t values near the minimum of the smooth potential, the condition for the applicability of this approximation can be written in the form

$$(\Phi_0 H_z/2\pi D^2 H_x^2)^2 \ll \Delta\lambda. \qquad (2.12)$$

This inequality enables us to estimate the width of that region of small angles $|\theta|<\theta^*$ in which the functional dependence $H(\theta)$ deviates from the dome shape. Using the explicit expression for $\Delta\lambda$ at $H\ll H_{cr}$ and (2.12), we find

$$\theta^*\approx 8\left(\frac{\pi}{2}\right)^{3/4}\frac{H_{cr}D^2}{\Phi_0}\left(\frac{H}{H_{cr}}\right)^{1/4}\exp\left\{-2\frac{H_{cr}}{H}\right\}. \qquad (2.13)$$

[This expression agrees with condition (1.7).]

At low temperatures ($H\gg H_{cr}$), over nearly the entire angular range the periodic potential in (2.11) has only a slight effect on the minimal eigenvalue λ and can be dealt with by first-order perturbation theory. The functional dependence $H(T,\theta)$ is then given by the expression

$$\tau\approx\frac{2\pi\xi^2 H\sin\theta}{\Phi_0}+\frac{\pi^2 d\xi H^2\cos^2\theta}{3\Phi_0}$$
$$+\frac{2\xi^2}{dl}\left\{1-\exp\left[-\frac{\pi D^2}{2\Phi_0}H\frac{\cos^2\theta}{\sin\theta}\right]\right\}. \qquad (2.14)$$

The condition for the applicability of perturbation theory is determined by the inequality $H\gg H_{cr}$. Equation (2.14) takes a particularly simple form at small angles, with

$$\frac{\sin\theta}{\cos^2\theta}\leqslant\frac{\pi D^2}{2\Phi_0}H, \qquad (2.15)$$

where it is sufficient to consider only the first two terms in (2.14). In this case, using the low-temperature asymptotic behavior[14] for the field $H_\parallel(T)$,

$$H_\parallel(T)=\frac{3^{1/2}\Phi_0}{\pi d\xi}\left(\frac{\tilde{T}_c-T}{T_c}\right)^{1/2},\quad \tilde{T}_c=T_c\left(1-\frac{2\xi^2}{dl}\right), \qquad (2.16)$$

and the customary expression for $H_\perp(T)$, we can put Eq. (2.14) in a form similar to that of Tinkham's formula[7]:

$$\left[\frac{H\cos\theta}{H_\parallel(T)}\right]^2+\frac{H\sin\theta}{H_\perp(T)}\frac{T_c-T}{\tilde{T}_c-T}=1. \qquad (2.17)$$

Using this equation, as in the case of the ordinary Tinkham formula, we can construct the angular dependence $H(\theta)$ knowing only the experimental values of $H_\perp(T)$ and $H_\parallel(T)$. The temperature $\tilde{T}_c$ is found from the experimental data by linearly extrapolating the low-temperature part of the $H_\parallel{}^2(T)$ dependence to $H=0$. Expression (2.17) evidently describes a decrease in the critical field with angle which is more rapid than that described by the Tinkham formula. The reason lies in the difference between the extrapolation temperature $\tilde{T}_c$ and T_c, which is in turn a consequence of the discrete nature of the structure of the superlattice and the distinctive proximity effect in a system of weakly coupled layers.[14]

3. STRUCTURAL FEATURES IN THE $H(\theta)$ DEPENDENCE IN A SUPERLATTICE WITH A SLAB INHOMOGENEITY

We showed above that the derivative $\beta(H)$ is finite in an ideal superlattice. In weak fields, where the size of the superconducting nucleating region, $l_H=(H_{cr}/2H)^{1/2}D$, is large in comparison with the period D of the superlattice, the values of $\beta(H)$ become small, since a large nucleating region is only slightly sensitive to the discrete nature of the structure of the superlattice. Under these conditions, the behavior $\beta(H)$ may be substantially affected by the irregularity of the superlattice, which leads to pinning near a defect of a superconducting nucleating region spanning a large number of layers.

Let us examine how $\beta(H)$ is affected by a very simple irregularity: a single "defective" metal layer, with a thickness which deviates from the nominal thickness by an amount $\Delta d\ll d$. The parameters D and l are assumed to remain constant. To calculate $\beta(H)$ we need to again find the spectrum of $\tau_\perp(H,k)$ values from Eq. (2.3), this time incorporating the inhomogeneity of the superlattice layers: $d(n)=d+\Delta d\delta_{n,0}$.

Since the size of the nucleating region, l_H, is large in comparison with D, we can switch from n to the continuous variable $Z=nD$ and replace the finite differences by derivatives. To lowest order in the parameters $\Delta d/d$ and $(D/l_H)^2$ we find the following equation from (2.3):

$$-\frac{d^2\psi}{dZ^2}+\left\{\left(\frac{Z-\Phi_0 k/2\pi H}{l_H^2}\right)^2\right.$$
$$\left.-\frac{dl}{D^2}\frac{\tau_\perp}{\xi^2}\right\}\left[1+\frac{\Delta d}{d}D\delta(Z)\right]\psi=0. \qquad (3.1)$$

The term with the δ-function is a small perturbation. To first order in the parameter $\Delta d/d$ we find from (3.1)

$$\tau_\perp(H,Z_0)=\tau_\perp^{(0)}(H)+\tau_\perp^{(1)}(H,Z_0),$$
$$\tau_\perp^{(0)}(H)=\frac{\pi D^2 H_{cr}}{\Phi_0}\frac{2\pi\xi^2 H}{\Phi_0},\quad \tau_\perp^{(1)}=\tau_\perp^{(0)}\frac{\Delta d\, D}{2\pi l_H d}$$
$$\cdot\left[\left(\frac{Z_0}{l_H}\right)^2-1\right]\exp\left\{-\frac{Z_0^2}{l_H^2}\right\} \qquad (3.2)$$

(for convenience in writing these expressions, we have introduced the coordinate of the center of the nucleating region, $Z_0 = \Phi_0 k/2\pi H$).

It can be seen from (3.2) that either a thick ($\Delta d > 0$) or a thin ($\Delta d < 0$) defective layer will result in the localization of a nucleating region. In the former case the center of the nucleating region "settles" at the inhomogeneity, while in the latter case it settles at a distance $2^{1/2} l_H$ from it. Since the analysis is absolutely the same in the two cases, we set $\Delta d > 0$ for definiteness. From (3.2) and (1.6) we find

$$\beta(H) \approx -\frac{2^{1/2}\Phi_0}{\pi^{5/4} D^2}\left(\frac{\Delta d}{d}\right)^{1/2}\left(\frac{H}{H_{cr}}\right)^{5/4}. \qquad (3.3)$$

The functional dependence $\beta(H) \propto H^{5/4}$ holds for an arbitrary disruption of the periodicity of the superlattice due to deviations of the thicknesses d and s from their nominal values, provided only that the total thickness of the defective region remains significantly smaller than the dimension of the nucleating region, l_H. In the case of thin insulating interlayers there is an extremely high probability for the formation of a defect consisting of two neighboring superconducting layers which "coalesce" because of a hole in the interlayer. For such a defect we have

$$\beta(H) = -(3^{3/2}/e\pi^{5/4})(\Phi_0/d^2)(H/H_{cr})^{5/4},$$

where we are assuming $S \ll d, D \approx d$.

A linear variation in the critical field as a function of the angle in the presence of defects of this sort should be observed in the region $\theta \lesssim \theta^*$:

$$\theta^* \sim \left|\frac{\Delta d}{d}\right|^{1/2} \frac{H_{cr} D^2}{\Phi_0}\left(\frac{H}{H_{cr}}\right)^{1/4}. \qquad (3.4)$$

It follows from (3.4) that when there is a defect in a superlattice the angular region near $\theta = 0$ in which $H(\theta)$ varies sharply becomes substantially larger than in the case of an ideal superlattice.

Comparing (3.3) and (2.10), we find the field interval for which the inhomogeneities dominate $\beta(H)$:

$$H \lesssim H_{cr}\left(1 + \frac{1}{4}\ln\left|\frac{d}{\Delta d}\right| + \frac{1}{2}\ln\frac{H_{cr}}{H}\right)^{-1}. \qquad (3.5)$$

According to (3.5), this region is $\sim 0.5 H_{cr}$ in the case $|\Delta d/d| \approx 0.1$.

4. EFFECT OF SUPERLATTICE BOUNDARIES ON $H(\theta)$

The boundary of a sample, like a slight inhomogeneity in the volume of a superlattice, leads to qualitative changes in $\beta(H)$ only in the limit of weak fields, where we have $l_H \gg D$. We first consider a single ideal superlattice boundary formed by decoupling the layers $n = 0$ and $n = -1$. In this case, Eq. (2.3) holds for all $n \geqslant 1$. For $n = 1$, the finite-difference operator in (2.3) should be replaced by the first-order finite-difference $\Delta^1 \psi_1 = \psi_1 - \psi_0$. When we switch to the continuous variable $Z = nD$ in (2.3), we find the boundary condition $\psi'(0) = 0$. The problem of determining the $\tau_\perp(H,k)$ spectrum thus reduces to the well-studied problem[7] of a surface superconductivity. In applying (1.6) to this case, we would like to put it in a dimensionless form, making use of the characteristic values H and k. We can then use the numerical results of Thompson.[12] Taking this approach, we find

$$\beta(H) = -1.30(\Phi_0/\pi D^2 H_{cr})H. \qquad (4.1)$$

The deviation from linearity in (4.1) [a positive curvature in $\beta(H)$], due to the discrete nature of the structure of the superlattice, arises in fields

$$H \gtrsim H_{cr}/(1 + {}^1\!/_4 \ln(\Phi_0/D^2 H_{cr})). \qquad (4.2)$$

In fields satisfying (4.2), the functional dependence $\beta(H)$ in the case of a surface conductivity is not qualitatively different from that shown in Fig. 1. In particular, the function $\beta(H)$ tends in the limit $H/H_{cr} \to \infty$ toward the value $-3\Phi_0/\pi d^2$, which corresponds to a thin film. The only difference is that the correction proportional to $(H_{cr}/H)^4$ is half the value given in (2.8).

In a superlattice of finite thickness $\mathscr{L}$, the field H_{cr} is supplemented by another characteristic field, the vortex entry field[7]

$$H_V \approx 1.6(D/\mathscr{L})^2 H_{cr} \ll H_{cr}.$$

Changes occur in expression (4.1) as H is reduced to a level on the order of H_V: $\beta(H)$ vanishes[12] as the square root of the field strength as $H \to H_V \pm 0$ and tends toward the nonzero value

$$\beta(0) = -\frac{3\Phi_0}{\pi\mathscr{L}^2}\frac{\Phi_0}{\pi D^2 H_{cr}} \qquad (4.3)$$

as $H \to 0$ (Fig. 2a). This value is of course far smaller than the value of β in a strong field, $\beta(0)/\beta(\infty) = (d/\mathscr{L})^2 \pi D^2 H_{cr}/\Phi_0 \ll 1$.

If a superlattice borders a normal metal, there is no surface superconductivity. In this case the nucleating region lies in the interior of the superlattice. Because of the finite size of the sample, the decrease in the field which results from the exponential decay of $\beta(H)$ [see (2.10)] in the region $H \lesssim H_V$ causes $\beta(H)$ to grow to a value on the order of that in (4.3) (Fig. 2b). A Nb/Ta superlattice with normal outer layers was studied in Ref. 19. In the case of a weak interlayer coupling, the behavior $\beta(H)$ for $H \ll H_{cr}$ is complicated, regardless of the type of interlayer (insulating or normal). The angular dependence for a sample with $D = 780$ Å and $d = 290$ Å is indeed free of the behavior (4.1), characteristic

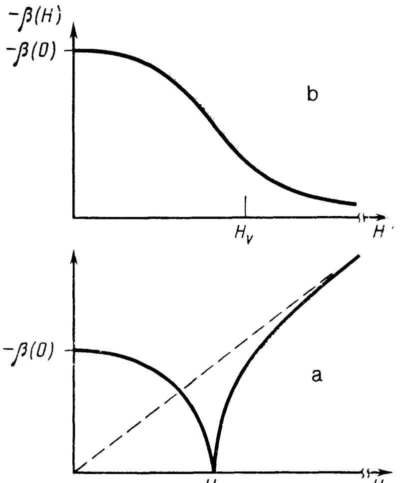

FIG. 2. Schematic plots of $\beta(H)$ for a bounded superlattice sample of thickness $\mathscr{L}$ in weak fields ($H \ll H_{cr}$). a—Ideal boundaries [the value of $\beta(0)$ is given by (4.3); the dashed line shows (4.1); H_V is the vortex entry field]; b—case of a superlattice bounded by layers of a normal metal.

of a surface superconductor. However, the deviations from the Lawrence-Doniach formula caused by the finite thickness of the sample, $\mathscr{L} = 5200$ Å, are also indistinguishable on the plots shown in Ref. 19. According to (4.3), the value of β in the limit $H \to 0$ would be only ~ 1 Oe/deg.

5. $H(\theta)$ FOR A SANDWICH OF TWO IDENTICAL FILMS

To find substantial differences from the behavior of the critical field for an isolated thin film we need go no further than a system consisting of only two identical metal layers with a Josephson junction between them. An analog of the crossover phenomenon characteristic of a superlattice arises in such a system. In a weak field,[1)] with

$$H < H_{cr}, \quad H_{cr} = 2^{1/2}\Phi_0/\pi d^{3/2} l^{1/2}, \tag{5.1}$$

the nucleating region spans the two metal layers, and its center is at the middle of the sandwich. In a strong field ($H > H_{cr}$) the nucleating region is localized in one of the layers. Correspondingly, the functional dependence $H_\parallel{}^2(T)$ has a characteristic positive curvature near values $H \sim H_{cr}$. The crossover is also manifested in the angular dependence of the critical field. For a sandwich one can find an analytic functional dependence $\beta(H)$ over the entire field range:

$$\beta(H) = -\frac{3\Phi_0}{4\pi d^2}\left[1 - \left(\frac{H}{H_{cr}}\right)^2\right]^{1/2}, \quad H < H_{cr}, \tag{5.2}$$

$$\beta(H) = -\frac{3\Phi_0}{\pi d^2}\left[1 - \left(\frac{H_{cr}}{H}\right)^4\right]^{1/2}\left[1 + 3\left(\frac{H_{cr}}{H}\right)^4\right]^{-1}, \quad H > H_{cr}. \tag{5.3}$$

The values of $\beta(H)$ at $H = 0$ and as $H \to \infty$ are the same as those for isolated films of thicknesses $2d$ and d, respectively, reflecting the change in the structure of the nucleating region in a sandwich as the field increases, as mentioned above.

In very weak fields, $H \ll H_{cr}$, the $H(\theta)$ dependence is described by Tinkham's formula. Under the condition $H \gg H_{cr}$, the angular dependence of the critical field is determined by (2.14), provided that the parameter $2\xi^2/dl$ in the last term is halved.

In strong fields, no qualitative differences are found in the behavior $\beta(H)$ and $H(\theta)$ for a sandwich and for a superlattice.

CONCLUSION

The upper critical field of a superlattice can be found for an arbitrary temperature and for an arbitrary orientation of **H** with respect to the superlattice layers, even on the basis of the Ginzburg-Landau equations, through numerical solution of these equations. Correspondingly, a calculation of this sort is meaningful only for specific experimental conditions. Analytic methods make possible substantial progress in the study of the behavior $H(T)$ in the case of two special orientations of **H**, parallel and perpendicular to the layers. These cases have been the subject of most of the studies (e.g., Refs. 14 and 20–24).

Analytic functions $H(\theta)$ can be constructed over broad angular regions if the problem has a small parameter. In weak fields, the small parameter is the ratio $H/H_{cr} \ll 1$, which makes it possible to derive the Lawrence-Doniach formula.[8] That formula, however, can not be used at small angles [see (2.13)]. In the case of a strong field, the small parameter is[17] $H_{cr}/H \ll 1$. It can be seen from (2.14) and (2.16) that in the latter case the expression for $H(\theta)$ differs from Tinkham's formula by terms of order $(T_c - \tilde{T}_c)/(\tilde{T}_c - T)$, which are proportional to the strength of the coupling between layers. At relatively small angles (2.15), expression (2.14) reduces to (2.17), which is similar to Tinkham's equation. Expression (2.17) can be used to construct the angular dependence $H(\theta)$ on the basis of no more than the experimental values of $H_\perp(T)$ and $H_\parallel(T)$. The differences between the expression derived here for $H(\theta)$ and the Lawrence-Doniach and Tinkham relations stem from the correct account of both the discrete structure of the superlattice and of the finite force of the coupling between individual layers in the superlattice.

The adiabatic method developed in the present paper can be used along with a different small parameter, the angle θ. Correspondingly, it becomes possible to determine the behavior $H(\theta)$ at small angles for arbitrary values of H/H_{cr}. We have used this method for an ideal structure to find the deviations [see (2.8) and (2.10)] from these other formulas for small angles. A more important point, however, is that the derivative $\partial^2\tau_\perp/\partial k^2$, which determines the value of $\beta = \partial H/\partial|\theta|\,|_{\theta\to 0}$, is very sensitive to irregularities of the superlattice at weak magnetic fields. In particular, we showed in Sec. 3 of this paper that a defective layer in a superlattice substantially increases the values of $\beta(H)$ for $H \ll H_{cr}$. Measurements of $H(\theta)$ at small angles can thus provide information on the irregularities of a superlattice.

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[1)]For brevity we are restricting the discussion to the case $s \ll d$.

[1]M. R. Beasley, in Proceedings of the NATO Advanced Study Institute, Vol. 109, Percolation, Localization, and Superconductivity, Academic, Orlando, 1984, p. 115.

[2]I. Banerjee and I. K. Schuller, J. Low Temp. Phys. **54**, 501 (1984).

[3]G. Deutscher and O. Entin-Wohlman, Phys. Rev. B **17**, 1249 (1978).

[4]L. N. Bulaevskii, Usp. Fiz. Nauk **116**, 449 (1975) [Sov. Phys. Usp. **18**, 514 (1975)].

[5]D. E. Prober, R. E. Schwall, and M. R. Beasley, Phys. Rev. B **21**, 2717 (1980).

[6]R. V. Coleman, G. K. Eiserman, S. J. Hillenius, A. T. Mitchell, and J. L. Vicent, Phys. Rev. B **27**, 125 (1983).

[7]D. Saint-James, G. Sarma, and E. J. Thomas, *Type II Superconductivity*, Pergamon, New York, 1969.

[8]W. Lawrence and S. Doniach, in: Proceedings of the Twelfth International Conference on Low-Temperature Physics, Kyoto-Tokyo, Academic, New York, 1971, p. 361.

[9]V. L. Tavozhnyanskii, V. G. Cherkasova, and N. Ya. Fogel', Zh. Eksp. Teor. Fiz. **93**, 1384 (1987) [Sov. Phys. JETP **66**, 787 (1987)].

[10]E. V. Minenko and I. O. Kulik, Fiz. Nizk. Temp. **5**, 1237 (1979) [Sov. J. Low Temp. Phys. **5**, 583 (1979)].

[11]L. D. Landau and E. M. Lifshitz, *Kvantovaya mekhanika*, Nauka, Moscow, 1974, p. 365 (*Quantum Mechanics*, Pergamon, 1977).

[12]R. S. Thompson, Zh. Eksp. Teor. Fiz. **69**, 2249 (1975) [Sov. Phys. JETP **42**, 1144 (1975)].

[13]A. V. Svidzinskii, *Prostranstvenno-neodnorodnye teorii sverkhprovodimosti*, (*Spatially Nonuniform Problems in Superconductivity Theory*), Nauka, Moscow, 1982.

[14]L. I. Glazman, I. M. Dmitrenko, V. L. Tavozhnyanskii, N. Ya. Fogel', and V. G. Cherkasova, Zh. Eksp. Teor. Fiz. **92**, 1461 (1987) [Sov. Phys. JETP **65**, 821 (1987)].

[15]M. Abramowitz and I. A. Stegun (editors), *Spravochink po spetsial'nym funktsiyam*, Nauka, Moscow, 1979, p. 532 (*Handbook of Mathematical*

Functions, Dover, New York, 1964).
[16]L. N. Bulaevskii, Zh. Eksp. Teor. Fiz. **64**, 2241 (1973) [Sov. Phys. JETP **37**, 1133 (1973).
[17]M. Menon and G. B. Arnold, *Superlattices and Microstructures* **1**, 451 (1985).
[18]A. I. Ansel'm, *Vvedenie v teoriyu poluprovodnikov* (*Introduction to the Theory of Semiconductors*), Nauka, Moscow, 1978, p. 206.
[19]P. R. Broussard and T. H. Geballe, Phys. Rev. B **35**, 1664 (1987).
[20]J. Simonin, Phys. Rev. B **33**, 1700 (1986).
[21]K. R. Biagi, V. G. Kogan, and J. R. Clem, Phys. Rev. B **32**, 1700 (1985).
[22]V. G. Kogan, Phys. Rev. B **32**, 139 (1985).
[23]S. Takahashi and M. Tachiki, Phys. Rev. B **33**, 4620 (1986).
[24]S. Takahashi and M. Tachiki, Phys. Rev. B **34**, 3162 (1986).

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