Conformal dimension spectrum for lattice integrable models of magnets

S. V. Pokrovskiĭ and A. M. Tsvelik

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The dimensions of all the conformally invariant operators are found in the scaling limit for the lattice antiferromagnetic spin-1/2 XXZ chain. The operator algebra of the XXZ model is shown to be generated by a discrete set of exponentials involving the free massless boson field. Hence it is possible to compute the infrared asymptotic form of the multipoint correlations. The results for the XXZ chain are extended to "magnets" with larger symmetry groups $SU(n)$, $O(2n)$, E_n.

1. INTRODUCTION

Exact solutions of the two-dimensional statistical and 1 + 1-dimensional quantum lattice models have played an important role in the development of the theory of phase transitions (see Baxter's monograph[1] and the references therein). In particular, the spin-1/2 XXZ magnetic model can be solved exactly. Several quasi-one-dimensional magnetic materials to which this model applies are currently known.[1)] One can measure experimentally both the neutron scattering cross sections, which are related to the spin pair correlation function, and the Brillouin scattering, which yields information on the 4-point correlation functions.

Theoretical calculations of the correlation functions in the lattice model are difficult. Although a well-established procedure (quantum inverse scattering problem) is available for diagonalizing integrable Hamiltonians,[4–6] only a few pair correlation functions have been calculated thus far.[7,8]

Methods from conformally invariant field theory can be used to find the multipoint correlation functions for systems at a phase transition point. This approach is based on the principle of conformal invariance for critical fluctuations[9] and requires that the operators form a closed algebra.[10] The constraints on the correlation functions imposed by these assumptions take the form of a system of "conformal bootstrap" equations. The structure of the operator algebra is completely determined by the bootstrap conditions for the two-dimensional and 1 + 1-dimensional field theories, because the conformal group is infinite-dimensional.[11] Several exact conformal bootstrap solutions have recently been found which describe various kinds of universal critical behavior.[11–14]

In this paper we show that the conformally invariant Ashkin-Teller model can be used to find the correlation functions for the XXZ model in the infrared limit.[14] We also determine the operator algebra for several generalizations of the XXZ model.

In order to relate the lattice model in the continuous limit to the solution of the conformal bootstrap equations, we analyze the low-energy portion of the spectrum of the lattice Hamiltonian. It was shown in Refs. 15 and 16 that if the lattice Hamiltonian becomes conformally invariant in the continuous limit, then the low-energy portion of the spectrum and the structure of the eigenstates near the ground state must satisfy stringent constraints. Each eigenstate $|n\rangle$ must correspond to a conformally invariant operator $\hat{\Phi}_n$ in the field theory, while the formulas[15,16]

$$E_n=N\varepsilon_0+\frac{2\pi}{N}v_F(\Delta_n-c/12); \tag{1.1}$$

$$P_n=2k_np_F+\frac{2\pi}{N}S_n. \tag{1.2}$$

relate the energy and momentum of the state $|n\rangle$ to the scaling dimension Δ_n and spin S_n.[2)] Here $N\gg 1$ is the number of lattice sites in the quantum chain, ε_0 is the thermodynamic energy density of the ground state, v_F is the Fermi velocity, which depends on how the lattice Hamiltonian is normalized, p_F is the Fermi momentum, and k_n is an integer. The contribution $2k_np_F$ to the momentum in (1.2) is due to Umklapp processes from one edge of the Fermi band to the other.[8] The "conformally invariant" contributions to the energy and momentum are effects that arise due to the finite size of the system ($N\gg 1$).[15,16] The thermodynamic terms in (1.1), (1.2) contain a contribution from the ultraviolet degrees of freedom, while the corrections of order $1/N$ are due to the infrared degrees of freedom. The coefficient c determines the correction to the ground-state energy and is an important parameter in the conformal theory[16]; it is equal to the central charge of the conformal algebra.[11] We note that the corrections of higher order in $1/N$ and $1/\ln N$ which have not been included in (1.1) can be interpreted as scaling corrections.[15]

An alternative method was proposed in Refs. 16 and 17, in which the central charge is calculated from the low-temperature specific heat C_v of the chain (per single site) according to the formula

$$C_V\doteq\frac{\pi c}{3v_F}T, \tag{1.3}$$

where T is the absolute temperature.

We stress that the scale dimensions Δ_n and the spins S_n cannot be arbitrary in a conformally invariant theory. The spectrum of conformal dimensions

$$\Delta_n{}^{(\pm)}={}^1/_2(\Delta_n\pm S_n) \tag{1.4}$$

splits up into series corresponding to irreducible representations (modules) of the conformal algebra,[11] the dimensions in each series forming an arithmetic progression. The minimum dimension in each series and the corresponding conformally invariant operator are called "primary," while all subsequent operators in the module are "secondary." The number of secondary representations with dimension $\Delta_d^{(\pm)}=\Delta_0^{(\pm)}+d^{(\pm)}$, $d^{(\pm)}$ a nonnegative integer, is equal to the number of partitions of $d^{(\pm)}$ as a sum of nonnegative integers. The spectrum of the primary dimensions is not specified by the requirement of conformal invariance but must be found by invoking the self-consistency conditions for the bootstrap equations.[11]

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The conformal dimension spectrum has been calculated completely for the Ising lattice model.[18] In Sec. 2 below, we use the Bethe Ansatz[19] to compute the spectrum for a spin-1/2 *XXZ* antiferromagnet described by the Hamiltonian

$$H=\frac{J}{2}\sum_{n=1}^{N}(\sigma_n^1\sigma_{n+1}^1+\sigma_n^2\sigma_{n+1}^2+\cos\gamma\sigma_n^3\sigma_{n+1}^3), \tag{1.5}$$

where the Pauli matrices σ_n^i act on the nth site, and γ is the anisotropy parameter. The primary dimensions found below are completely specified by two integer parameters k and m:[3)]

$$\Delta_{k,m}^{(\pm)}={}^1/_2(k\zeta\pm m/2\zeta)^2,\quad \zeta=[2(1-\gamma/\pi)]^{-1/2}. \tag{1.6}$$

The *XXZ* model in the continuous limit is widely regarded as equivalent to the massless Thirring model

$$H=\int\left\{i\psi_1^+\frac{\partial}{\partial x}\psi_1-i\psi_2^+\frac{\partial}{\partial x}\psi_2+g:\psi_1^+\psi_1\psi_2^+\psi_2:\right\}dx, \tag{1.7}$$

where the symbol :: denotes normal ordering. The primary conformally invariant operators in the Thirring model are expressible as products of the fermion degrees of freedom: $:\psi_1^n\psi_2^m:$, $:\psi_1^n\psi_1^{+m}:$, etc. The bosonization formulas[20] can be used to express these operators in the form

$$V_{n,m}(z_+,z_-)=:\exp\left\{\frac{i}{2}[(\beta_+n+\beta_-n)\varphi^{(+)}(z_+)+(\beta_+m-\beta_-m)\varphi^{(-)}(z_-)]\right\}:,$$

$$\beta_\pm=\left(\frac{1\pm g}{1\mp g}\right)^{1/2},\quad \Delta_{n,m}^{(\pm)}=\frac{1}{8}(\beta_+n\pm\beta_-m)^2, \tag{1.8}$$

where $z_\pm=t\pm ix$ are the standard complex coordinates,[11] and $\varphi^{(\pm)}(z_\pm)$ are the holomorphic and antiholomorphic components of the free massless real boson field $\Phi(z_+,z_-)$. The fermions themselves coincide with the following operators:

$$\psi_1=V_{-1,-1},\quad \psi_2=V_{1,-1},\quad \psi_1^+=V_{-1,1},\quad \psi_2^+=V_{1,1}. \tag{1.9}$$

The degrees of the fermions coincide with the operators (1.8) for n, m of equal parity. A comparison of the dimensions in (1.6) and (1.8) reveals that the primary operators (1.8) correspond to the *XXZ* model if $n=2k$, with $\beta_+=\zeta$.

The operator algebra in the conformally invariant Ashkin-Teller model[14] contains all the operators $V_{n,m}$ (Refs. 14, 21, 22). A sector with n, m of the same parity corresponds to the Thirring model, while a sector with n even corresponds to the *XXZ* model. The two models are equivalent in sectors with n, m both even.

The generalized *XXZ* models considered below can also be described using an r-component massless boson field theory. In this theory the central charge c is equal to r (Ref. 11). As an illustration, consider the integrable models with the Hamiltonians[23,24]

$$H=J\sum_{n=1}^{N}\left\{\sum_{i\neq j}X_n^{ij}X_{n+1}^{ji}+\cos\gamma\sum_i X_n^{ii}X_{n+1}^{ii}+i\sin\gamma\sum\mu_{ij}X_n^{ii}X_{n+1}^{jj}\right\}, \tag{1.10}$$

$$(X^{ij})_{\alpha\beta}=\delta_{i\alpha}\delta_{j\beta},\quad \mu_{ij}=\mathrm{sign}(i-j)+\frac{i-j}{r+1},\quad i,j=1,\ldots,r+1$$

This series contains the *XXZ* model ($r=1$). The isotropic Hamiltonians in (1.10) are invariant under the group $SU(r+1)$, provided $\gamma=0$ (Refs. 25, 26). The primary operators in this model are specified by an r-dimensional vector α in the lattice generated by the simple roots of the Lie algebra of $SU(r+1)$ together with a vector ω in the dual lattice (ω is the highest weight of some irreducible representation of the algebra):

$$V_{\alpha,\omega}(z_+,z_-)=:\exp\left\{i\left(\frac{\alpha}{2\sqrt{2}\zeta}+\sqrt{2}\zeta\omega\right)\varphi^{(+)}(z_+)+i\left(\sqrt{2}\zeta\omega-\frac{\alpha}{2\sqrt{2}\zeta}\right)\varphi^{(-)}(z_-)\right\}:, \tag{1.11}$$

$$\zeta=\left[2\left(1-\frac{\gamma}{\pi}\right)\right]^{-1/2},\quad \Delta_{\alpha,\omega}=\frac{1}{2}\left(\sqrt{2}\zeta\omega\pm\frac{\alpha}{2\sqrt{2}\zeta}\right)^2,$$

where $\varphi^{(\pm)}(z_\pm)$ denotes the holomorphic and antiholomorphic components of the vector boson field $\Phi(z_+,z_-)$, respectively.

The rest of our discussion proceeds as follows. Section 2 is devoted to a detailed analysis of weakly excited states in the *XXZ* model, and the conformal dimensions are calculated. The classification of solvable models is considered in Sec. 3, where we are particularly interested in models with an integral central charge, for which we calculate the dimension spectrum. The results are briefly discussed in Sec. 4.

2. DIMENSION SPECTRUM AND CLASSIFICATION OF STATES FOR AN *XXZ* MAGNET

The Hamiltonian for the *XXZ* model commutes with the spin operator

$$S^z=\sum_{n=1}^{N}\sigma_n^3. \tag{2.1}$$

The Hamiltonian (1) thus remains integrable if the term hS_z is added to it, where h is the magnetic field. In what follows we will analyze the energy and momentum spectra for an *XXZ* magnet in a moderately strong magnetic field $1\gg h/J\gg 1/N$.

The eigenstates of the Hamiltonian for the *XXZ* model can be found by the quantum inverse scattering method.[5] If the total spin S^z is specified, the state is parametrized by a set of $M=N/2-S^z$ rapidities λ_α ($\alpha=1,\ldots,M$) which satisfy the system of Bethe equations[19]

$$Z(\lambda_\alpha)=I_\alpha/N; \tag{2.2}$$

$$Z(\lambda)=\frac{1}{2\pi}\Theta_1(\lambda)-\frac{1}{2\pi N}\sum_{\beta=1}^{M}\Theta_2(\lambda-\lambda_\beta), \tag{2.3}$$

where

$$\Theta_n(\lambda)=\frac{1}{2i}\ln\frac{\mathrm{sh}\,\gamma(\lambda-in)}{\mathrm{sh}\,\gamma(\lambda+in)}. \tag{2.4}$$

The numbers I_α are half-integral (integral) for M even (odd). The state is completely specified by the number M and the set of quantum numbers $\{I_\alpha\}$; its energy and momentum are given by

$$P=\sum_{\alpha=1}^{M}\Theta_1(\lambda_\alpha), \tag{2.5}$$

$$E=\sum_{\alpha=1}^{M} t(\lambda_\alpha),\quad t(\lambda)=\frac{\sin\gamma}{\gamma}\Theta_1'(\lambda)-h. \tag{2.6}$$

In a sector with a specified M, the quantum numbers for the state with minimum energy are given by

$$I_\alpha=(1-M)/2+\alpha-1, \tag{2.7}$$

We will refer to such a distribution as a symmetrically filled Fermi band. Among all the possible ground states corresponding to different M, there exists a value $M=M_0(\gamma,h)$ for which the energy is a minimum; moreover, $M_0(\gamma,0)=N/2$.

The ground-state solution of the Bethe equations is specified by a set of real-valued rapidities λ_α (Ref. 19), while the excited states are described by complex-valued rapidities. For $h\gg J/N(N\gg 1)$, the complex rapidities can be grouped together into strings. Excitations of the string type lie above the ground state by an energy comparable in order of magnitude to the strength of the magnetic field.[27] In other words, the string excitations lie in the high-energy part of the spectrum and may be neglected in the subsequent analysis.

Excitations with real-valued rapidities may be classified as follows. In an excited state, some of the numbers I_α in the distribution (2.7) may lie outside the Fermi band (Fig. 1). We write I_h for the coordinates of the holes thus produced in the Fermi band, and I_p for the coordinates of the particles lying outside the band. In terms of the configurations of the numbers I_α for weakly excited states, the particles and holes are localized near the edges of the Fermi band; we may therefore segregate them into "left" and "right" populations whose interaction is negligible. Hole/particle balance is observed only globally; the number of right (left) particles is not necessarily equal to the number of right (left) holes. Let k be the difference between the number of right particles and holes (Fig. 2a). We will show that for states with a specified M and k, the distribution of quantum states minimizing the energy is given by

$$I_\alpha=k+(1-M)/2+\alpha-1. \tag{2.8}$$

We will call this distribution a "shifted Fermi band" (Fig. 2b). Excitations above the state (2.8) are readily describable in terms of holes and particles above the shifted Fermi band (Fig. 2c). States in a sector with fixed M and k can be conveniently specified by means of the half-integral quantum numbers $J_p^{(\pm)}$ and $J_h^{(\pm)}$, defined as the distances of the right (left) holes and particles from the right- and left-hand edges $\pm M/2+k$ of the Fermi band.

An expression for the momentum in terms of the quantum numbers of the state can be found by adding the Bethe equations (2.3). Since the functions (2.4) are odd, we obtain

$$P=\frac{2\pi}{N}\sum_{\alpha=1}^{M} I_\alpha=\frac{2\pi}{N}(Mk+d^{(+)}-d^{(-)}), \tag{2.9}$$

where

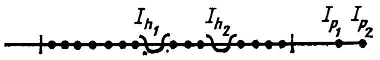

FIG. 1.

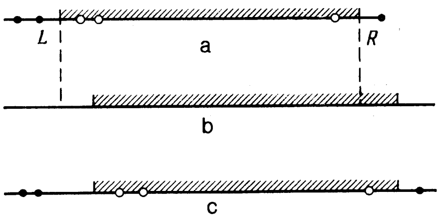

FIG. 2.

$$d^{(\pm)}=\sum_p J_p^{(\pm)}+\sum_h J_h^{(\pm)}. \tag{2.10}$$

In terms of the Fermi momentum $p_F=\pi M_0/N$ and quantum number $m=M-M_0$, we find

$$P=2kp_F+\frac{2\pi}{N}(mk+d^{(+)}-d^{(-)}). \tag{2.11}$$

The energy is somewhat harder to calculate. One must determine how the energy of states with a symmetrically filled Fermi band depends on the number of particles N. Using the Euler-Maclaurin formula, we may replace the summations in (2.4) and (2.6) by an integral,

$$\sum_{\alpha=1}^{M} F(\lambda_\alpha)=\sum_{\alpha=1}^{M} F(\Lambda(Z_\alpha))=\int_{-M/2N}^{M/2N} F(\lambda(Z))\,dZ$$
$$-\frac{1}{24N}\frac{d}{dZ}F(\Lambda(Z))\Big|_{-M/2N}^{M/2N}+Q(N^{-3}),\quad Z_\alpha=\frac{I_\alpha}{N}, \tag{2.12}$$

where the function $\Lambda(Z)$ is inverse to $Z(\lambda)$. A similar procedure was employed in Ref. 28.

Differentiating (2.3) with respect to λ and using (2.12), we find that the density distribution

$$\sigma(\lambda)=dZ/d\lambda \tag{2.13}$$

of the roots of Eq. (2.3) satisfies the integral equation

$$\hat{K}(Q)\sigma(\lambda)=\frac{1}{2\pi}\Theta_1'(\lambda)$$
$$-\frac{1}{48\pi N^2}\left\{\frac{\Theta_2''(\lambda-Q)}{\sigma(Q)}-\frac{\Theta_2''(\lambda+Q)}{\sigma(-Q)}\right\}, \tag{2.14}$$

where $Q=\Lambda(M/2N)$ is the edge of the Fermi band. The linear integral operator $\hat{K}(Q)$ acts on a function f as follows:

$$\hat{K}(Q)f(\lambda)=f(\lambda)+\int_{-Q}^{Q}\Theta_2'(\lambda-\mu)f(\mu)\frac{d\mu}{2\pi}. \tag{2.15}$$

Using Eq. (2.12), we can rewrite (2.6) as

$$\frac{E}{N}=\int_{-Q}^{Q} t(\lambda)\sigma(\lambda)\,d\lambda-\frac{1}{24N}\frac{t'(\lambda)}{\sigma(\lambda)}\Big|_{-Q}^{Q}$$

$$=\langle t,\sigma\rangle_Q-\frac{1}{24N}\frac{t'(\lambda)}{\sigma(\lambda)}\Big|_{-Q}^{Q}. \tag{2.16}$$

where we have introduced the inner product

$$\langle f,g\rangle_Q=\int_{-Q}^{Q} f(\lambda)g(\lambda)\,d\lambda, \tag{2.17}$$

and by definition (2.13),

$$\frac{M}{N}=\int_{-Q}^{Q}\sigma(\lambda)\,d\lambda\equiv\langle\sigma,1\rangle_Q. \quad (2.18)$$

Equations (2.14)–(2.18) determine the dependence $E(M)$ implicitly. The ground state is given by the condition $dE/dM=0$. It is convenient to work with new functions $\epsilon(\lambda)$ and $\zeta(\lambda)$ satisfying the integral equations

$$\hat{K}(Q)\varepsilon(\lambda)=t(\lambda), \quad (2.19)$$

$$\hat{K}(Q)\zeta(\lambda)=1. \quad (2.20)$$

Combining (2.19) with (2.14) and (2.16), we get the expression

$$\frac{E}{N}=\frac{1}{2\pi}\langle\varepsilon,\Theta_1'\rangle_Q-\frac{1}{24N^2}\frac{\varepsilon'(\lambda)}{\sigma(\lambda)}\Big|_{-Q}^{Q} \quad (2.21)$$

for the energy in terms of $\varepsilon(\lambda)$; in the same way, we obtain

$$\frac{M}{N}=\frac{1}{2\pi}\langle\zeta,\Theta_1'\rangle_Q-\frac{1}{24N^2}\frac{\zeta'(\lambda)}{\sigma(\lambda)}\Big|_{-Q}^{Q}. \quad (2.22)$$

In deriving Eqs. (2.21) and (2.22) we have used the fact that the operator $\hat{K}(Q)$ is Hermitian.

In calculating the derivatives dE/dQ and dM/dQ we may neglect the terms of order $1/N^2$ in the Euler-Maclaurin expansions (2.21), (2.22). To treat the explicit dependence of the functions $\varepsilon(\lambda,Q)$ and $\zeta(\lambda,Q)$ on Q, one must differentiate Eqs. (2.19) and (2.20) with respect to Q.

The final expressions for these derivatives are as follows:

$$\frac{1}{N}\frac{dE}{dQ}=\varepsilon(\lambda)\sigma(\lambda)\big|_{-Q}^{Q}=2\varepsilon(Q)\sigma(Q), \quad (2.23)$$

$$\frac{1}{N}\frac{dM}{dQ}=\zeta(\lambda)\sigma(\lambda)\big|_{-Q}^{Q}=2\zeta(Q)\sigma(Q). \quad (2.24)$$

We have exploited the parity properties of the functions $\varepsilon(\lambda)$, $\sigma(\lambda)$, $\zeta(\lambda)$ in deriving these formulas. Minimum energy occurs at the value $Q=Q_0$ at which

$$\varepsilon(\lambda,Q)\big|_{\lambda=Q_0}=0. \quad (2.25)$$

The corresponding extremum $M_0=M(Q_0)$ need not be an integer. The integral values of M_0 for a specified anisotropy γ determine a discrete series of magnetic fields. The second derivative d^2E/dQ^2 at $Q=Q_0$ is easier to evaluate because of the equality

$$\frac{\partial\varepsilon}{\partial Q}(\lambda,Q)\big|_{Q=Q_0}=0.$$

We have

$$\frac{1}{N}\frac{d^2E}{dQ^2}\Big|_{Q=Q_0}=2\varepsilon'(Q_0)\sigma(Q_0). \quad (2.26)$$

According to (2.21) and (2.24)–(2.26), the energy of a state with a symmetrically filled Fermi band is given by

$$E(M)=N\varepsilon_0+\frac{2\pi}{N}v_F\left(\frac{m^2}{4\zeta_0^2}-\frac{1}{12}\right),\quad m=M-M_0, \quad (2.27)$$

as a function of M; here ε_0 is the ground-state energy density in the thermodynamic limit, and ζ_0 and the Fermi velocity v_F are given by

$$\zeta_0=\zeta(\lambda,Q_0)\big|_{\lambda=Q_0}, \quad (2.28a)$$

$$v_F=\frac{1}{2\pi}\left\{\frac{\varepsilon'(\lambda,Q_0)}{\sigma(\lambda,Q_0)}\right\}\Big|_{\lambda=Q_0}, \quad (2.28b)$$

$$\varepsilon_0=\frac{1}{2\pi}\langle\varepsilon,\Theta_1'\rangle_{Q_0}. \quad (2.28c)$$

The technique suggested in Ref. 29 can be used to find the energy of excited states lying in a sector with a specified M. Let the excited state be described by a hole-particle configuration $\{I_p^{(+)}, I_h^{(+)}, I_p^{(-)}, I_h^{(-)}\}$, where the superscripts + and − refer to left and right excitations, respectively. Writing $\{\tilde{\lambda}_\alpha\}$ for the set of roots of the Bethe equations (2.2), (2.3) in the excited state, we can rewrite the function (2.3) in the form

$$\tilde{Z}(\lambda)=\frac{1}{2\pi}\Theta_1(\lambda)-\frac{1}{2\pi N}\sum_{\alpha=1}^{M}\Theta_2(\lambda-\tilde{\lambda}_\alpha)+\frac{1}{2\pi N}\sum_{h=1}^{n}\Theta_2(\lambda-\tilde{\lambda}_h)-\frac{1}{2\pi N}\sum_{p=1}^{n}\Theta_2(\lambda-\tilde{\lambda}_p). \quad (2.29)$$

The shift in the rapidities in the Fermi band is given by the function $\delta\Lambda(\lambda)$:

$$\tilde{Z}(\lambda+\delta\Lambda(\lambda))=Z(\lambda); \quad (2.30)$$

$$\tilde{\lambda}_\alpha=\lambda_\alpha+\delta\Lambda(\lambda_\alpha). \quad (2.31)$$

The function $Z(\lambda)$ in Eq. (2.30) is for a symmetrically populated Fermi band. The quantity $\delta\Lambda$ is of order $1/N$, and in this approximation the rapidities for the holes and particles are equal to

$$\begin{aligned}\tilde{\lambda}_h^{(\pm)}&=\pm Q+\delta\Lambda(\pm Q)\mp\Delta I_h^{(\pm)}/N\sigma(\pm Q),\\ \tilde{\lambda}_p^{(\pm)}&=\pm Q+\delta\Lambda(\pm Q)\pm\Delta I_p^{(\pm)}/N\sigma(\pm Q),\end{aligned} \quad (2.32)$$

where $\Delta I_p^{(\pm)}$ and $\Delta I_h^{(\pm)}$ are the distances of the right (left) particles and holes from the right-hand (left-hand) edges of the Fermi band. We get a finite-difference equation for the function $\delta\Lambda(\lambda)$ by expanding (2.30) in powers of $\delta\Lambda$ and the small corrections to the particle rapidities. Using (2.12) to replace the sums by integrals (The Euler-Maclaurin corrections may be neglected), we obtain a nonlinear integral equation for $\delta\Lambda(\lambda)$ which can be solved by successive approximations:

$$\sigma(\lambda)\delta\Lambda(\lambda)=\frac{1}{N}\xi(\lambda)+\frac{1}{N^2}\eta(\lambda)+\ldots \quad (2.33)$$

The functions ξ and η satisfy the following integral equations:

$$\hat{K}(Q)\xi(\lambda)=\frac{k}{2\pi}[\Theta_2(\lambda-Q)-\Theta_2(\lambda+Q)]; \quad (2.34)$$

$$\hat{K}(Q)\eta(\lambda)=\frac{1}{2}\frac{\partial}{\partial\lambda}\left\{\hat{K}(Q)\left[\frac{\xi^2(\lambda)}{\sigma(\lambda)}\right]\right\}-\frac{1}{2\pi}\frac{\Theta_2''(\lambda-Q)}{\sigma(Q)}[k\xi(Q)+D^{(+)}]+\frac{1}{2\pi}\frac{\Theta_2''(\lambda+Q)}{\sigma(-Q)}[k\xi(-Q)+D^{(-)}], \quad (2.35)$$

where

$$D^{(\pm)}=\sum_{p=1}^{\frac{1}{2}(n\pm k)}\Delta I_p^{(\pm)}+\sum_{h=1}^{\frac{1}{2}(n\mp k)}\Delta I_h^{(\pm)}=\frac{k^2}{2}+d^{(\pm)}. \qquad (2.36)$$

The last equality in (2.36) holds for a shifted (as opposed to symmetrically filled) Fermi band; the numbers $d^{(\pm)}$ are given by (2.10).

To get an expression for the energy of the excited state, we express the energy in the form

$$\tilde{E}=\sum_{j=1}^{M}t(\tilde{\lambda}_j)+\sum_{p=1}^{n}t(\tilde{\lambda}_p)-\sum_{h=1}^{n}t(\tilde{\lambda}_h) \qquad (2.37)$$

and subtract (2.7) for the energy of a state with a symmetrically filled Fermi band. After expanding the resulting expression in powers of $1/N$, using Eqs. (2.31) and (2.32), we may use (2.12) to replace the sums by integrals, neglecting the Euler-Maclaurin corrections. We thus obtain the following expressions for $\delta E=\tilde{E}-E$ to first and second order:

$$\delta E^{(1)}=\langle t'(\lambda),\xi(\lambda)\rangle_Q+kt(\lambda)\,|_{-Q}^{Q}, \qquad (2.38)$$

$$\delta E^{(2)}=\langle t'(\lambda),\eta(\lambda)\rangle_Q+\frac{1}{2}\langle t''(\lambda),(\xi^2(\lambda)/\sigma(\lambda))\rangle_Q$$
$$+\frac{t'(Q)}{\sigma(Q)}[k\xi(Q)+D^{(+)}]-\frac{t'(-Q)}{\sigma(-Q)}[k\xi(-Q)+D^{(-)}]. \qquad (2.39)$$

Using the integral equations (2.34), (2.35) and recalling the relation

$$\xi(\lambda)+k=k\zeta(\lambda), \qquad (2.40)$$

we find the expressions

$$\delta E^{(1)}=k\varepsilon(\lambda)\zeta(\lambda)\,|_{-Q}^{Q}=0, \qquad (2.41)$$

$$N\delta E^{(2)}=\frac{1}{2}\left\{\frac{\varepsilon'(\lambda)}{\sigma(\lambda)}k^2\zeta^2(\lambda)-\varepsilon(\lambda)\frac{\partial}{\partial\lambda}\left[\frac{\xi^2(\lambda)}{\sigma(\lambda)}\right]\right\}\Big|_{-Q}^{Q}$$
$$+\frac{\varepsilon'(Q)}{\sigma(Q)}d^{(+)}-\frac{\varepsilon'(-Q)}{\sigma(-Q)}d^{(-)}. \qquad (2.42)$$

The first-order correction $\delta E^{(1)}$ (2.41) vanishes because the functions $\varepsilon(\lambda)$ and $\zeta(\lambda)$ are even. If we set $Q=Q_0$ in (2.42), the resulting error is of order $1/N$, and in this approximation the term in curly brackets on the right in (2.42) vanishes by Eq. (2.25). Thus,

$$\delta E^{(2)}=\frac{2\pi}{N}v_F\{k^2\zeta_0^2+d^{(+)}+d^{(-)}\}. \qquad (2.43)$$

The dimensions for the excited state can be found from (2.11), (2.26), and (2.43) together with (1.1), (1.2), (1.4); the result is[4)]

$$\Delta^{(\pm)}=\frac{1}{2}\left(k\zeta_0\pm\frac{m}{2\zeta_0}\right)^2+d^{(\pm)}. \qquad (2.44)$$

Since the $d^{(\pm)}$ are integers, it seems plausible that a sector of Bethe states with specified m, k should correspond to a conformal representation (module). Indeed, the multiplicity of a level with specified $d^{(\pm)}$ is equal to the number of partitions of the type (2.10). It is equal to the number of free secondary representations in the free fermion theory,[18] which in turn coincides with the level multiplicity in the general conformally invariant theory.[11]

We conclude this section with a formula for the value of ζ_0 at $h=0$; this corresponds to letting $Q_0\to\infty$ in the integral equation (2.20). If we neglect the dependence on the lower limit $-Q$ in (2.15) in evaluating $\zeta(\lambda+Q_0,Q_0)$, where $\lambda\ll Q_0$, the Fredholm equation (2.20) reduces to a Wiener-Hopf equation,[19] which can be solved to give

$$\zeta_0=[2(1-\gamma/\pi)]^{-1/2}. \qquad (2.45)$$

3. SOLVABLE MODELS WITH HIGHER DEGREES OF SYMMETRY

In this section we consider some multicomponent generalizations of the XXZ model. The model Hamiltonians are constructed from factored R-matrices. Let the matrix $R_{ab}(\lambda)$ act on the tensor product space $V_a\otimes V_b$, and assume that it satisfies the triangle equations[5)] (Refs. 5, 6); then we can construct a parametric family of transfer matrices

$$T(\lambda)=Tr_0\{R_{01}(\lambda)R_{02}(\lambda)\ldots R_{0N}(\lambda)\}. \qquad (3.1a)$$

The Hamiltonian and the momentum operator for the corresponding model are expressible in terms of the transfer matrix[1] as follows:

$$H=J\frac{d}{d\lambda}[\ln T(\lambda)]_{\lambda=0},\quad P=-i\ln[T(\lambda)]_{\lambda=0}. \qquad (3.1b)$$

The factored R-matrices are classified in terms of the representations of simple Lie groups.[30-33] A solution of the triangle equations is said to be invariant under the action of a group G if

$$R_{ab}(\lambda)=T_a(g)\otimes T_b(g)R_{ab}(\lambda)T_a^{-1}(g)\otimes T_b^{-1}(g), \qquad (3.2)$$
$$g\in G.$$

Here the representation of G on V_a and V_b is expressed in terms of the matrices T_a and T_b. The matrix $R_{ab}(\lambda)$ is a rational function of λ. Any representation V_a is obtainable as a symmetric power of a "fundamental quantum group" representation[33]: $V_a=[V_a]^p$. These fundamental representations are in general reducible in the ordinary group-theoretic sense. A complete list is given in Ref. 30, where it is also shown that any representation

$$v_a=v_a^{(1)}\oplus\ldots\oplus v_a^{(n)}$$

is uniquely characterized by the first term $v_a^{(1)}$ in the direct sum. If ω_a is its highest weight (we refer to ω_a as the highest weight of the fundamental representation of the quantum group), then the highest weight of the representation V_a is $\Omega_a=p\omega_a$.

Any rational solution of the triangle equations can be generalized to a trigonometric solution[6)] (Refs. 32, 34). The trigonometric R-matrix is no longer invariant under G, but only under the Cartan subgroup $(U(1))^r$. Nevertheless, the Bethe equations and the results for the energy and momentum as functions of the state rapidities can still be expressed in terms of the roots of the Lie algebra and the weights of the Lie-algebra representations. Let the Hamiltonian (3.1b) be built up from R-matrices which are invariant under a group G of rank r or, more generally, from the corresponding trigonometric R-matrices. The eigenvalues of the Hamiltonian are characterized by the set of rapidities λ_α^i, which satisfy the Bethe equations[30-33]

$$Z_i(\lambda_\alpha{}^i)=I_\alpha{}^i/N,\ i=1,2,\ldots,r,$$
$$\alpha=1,\ldots,M_i,\ M_1>M_2>\ldots>M_r, \tag{3.3}$$

where

$$Z_i(\lambda)=\frac{1}{2\pi}\varphi_i(\lambda)-\frac{1}{2\pi N}\sum_{j=1}^{r}\sum_{\beta=1}^{M_j}\Phi_{ij}(\lambda-\lambda_\beta{}^j). \tag{3.4}$$

Here φ_i and Φ_{ij} are expressible in terms of the functions (2.4):

$$\varphi_i(\lambda)=\Theta_{(\alpha_i,\alpha_a)}(\lambda),\quad \Phi_{ij}(\lambda)=\Theta_{(\alpha_i,\alpha_j)}(\lambda), \tag{3.5}$$

where the α_i are the simple roots of the Lie algebra. The energy and momentum are

$$P=\sum_{i=1}^{r}\sum_{\alpha=1}^{M_i}\varphi_i(\lambda_\alpha{}^i), \tag{3.6}$$

$$E=\sum_{i=1}^{r}\sum_{\alpha=1}^{M_i}t_i(\lambda_\alpha{}^i),\quad t_i(\lambda)=\varphi_i'(\lambda)-h_i, \tag{3.7}$$

where the "magnetic fields" h_i are conjugate to the integrals M_i of the motion.

Equations (3.3) and (3.7) can be used to calculate the low-temperature specific heat of the system, and hence by (1.3) its central charge. The central charge for an $SU(2)$-invariant magnet is related to the spin S by the formula $c=3S/(S+1)$ (Ref. 35). In the general case of a G-invariant magnet, whose spin variables take values in a representation space $V_a=[v_a]^p$ of a quantum group, the central charge is equal to

$$c=p\mathscr{D}/(p+C_A), \tag{3.8}$$

where $\mathscr{D}$ denotes the dimension of the group G and C_A is the value of the square of the Casimir operator in the adjoint representation. A proof of Eq. (3.8) in the general case will be published elsewhere.

It is known that for $p=1$ (i.e., for a fundamental representation)

$c=$ the r-rank of the Lie algebra

for any Lie algebra whose simple roots all have the same length (i.e., the algebras A_n, D_n, E_n). These models yield generalizations of the XXZ model, for which $c=1$. From the standpoint of the Bethe Ansatz, they are also quite similar: the vacuum is again filled with real-valued rapidities. We may therefore employ the formalism developed in Sec. 3 to calculate the dimensions. We will discuss these calculations only briefly, pointing out where they differ significantly from the ones in Sec. 2.

Thus, if the simple roots of the Lie algebra all have the same length, then in a sector with specified quantum numbers $M_1>M_2>\ldots>M_r$ the ground state corresponds to a Fermi band which is symmetrically filled by rapidities of each type $i=1,2,\ldots,r$:

$$I_\alpha{}^i=(1-M_i)/2+\alpha-1. \tag{3.9}$$

The excited states are described by the quantum numbers k_i (the shift in the ith Fermi band) and by the distances $J_p^{i(\pm)}$, $J_h^{i(\pm)}$ from the edges of the band.

Adding all the Bethe equations (3.2), we readily obtain

$$P=\frac{2\pi}{N}\sum_{i=1}^{r}\sum_{\alpha=1}^{M_i}I_\alpha{}^i=2\sum_{i=1}^{r}k_ip_i{}^F+\frac{2\pi}{N}\sum_{i=1}^{r}(m_ik_i+d_i{}^{(+)}-d_i{}^{(-)}), \tag{3.10}$$

where

$$d_i{}^{(\pm)}=\sum_{p=1}^{n_i}J_p{}^{i(\pm)}+\sum_{h=1}^{n_i}J_h{}^{i(\pm)}; \tag{3.11}$$

$$m_i=M_i-M_i{}^{(0)},\quad p_i{}^F=\pi M_i{}^{(0)}/N. \tag{3.12}$$

Here the $M_i^{(-)}$ are the magnetic quantum numbers corresponding to minimum energy.

In the continuous limit the state is described by densities $\sigma_i(\lambda)$, $i=1,2,\ldots,r$ satisfying a system of integral equations

$$(\hat{K}\sigma(\lambda))_i=\frac{1}{2\pi}\varphi_i'(\lambda). \tag{3.13}$$

The operator $\hat{K}$ acts on the vector-valued function f_i by the formula

$$(\hat{K}f)_i(\lambda)=f_i(\lambda)+\sum_{j=1}^{r}\int_{-Q_j}^{Q_j}\Phi_{ij}'(\lambda-\mu)f_j(\mu)\frac{d\mu}{2\pi}. \tag{3.14}$$

The energy and the numbers M_i can be expressed in terms of the densities $\sigma_i(\lambda)$ by means of the inner product

$$\langle f,g\rangle_{\{Q\}}=\sum_{j=1}^{r}\int_{-Q_j}^{Q_j}f_j(\lambda)g_j(\lambda)d\lambda \tag{3.15}$$

One obtains

$$E/N=\langle t,\ \sigma\rangle_{\{Q\}}, \tag{3.16}$$

$$M_i/N=\langle\sigma,\ e^i\rangle_{\{Q\}},\quad (e^i)_k=\delta_k{}^i. \tag{3.17}$$

It is helpful to rewrite expressions (3.16) and (3.17) in terms of the "dressed" densities:

$$(\hat{K}\varepsilon)_i(\lambda)=t_i(\lambda), \tag{3.18}$$

$$(\hat{K}\zeta^j)_i(\lambda)=\delta_i{}^j. \tag{3.19}$$

Using (3.18), (3.19) we can write

$$\frac{1}{N}E=\frac{1}{2\pi}\langle\varepsilon,\varphi\rangle_{\{Q\}}, \tag{3.20}$$

$$\frac{M_i}{N}=\frac{1}{2\pi}\langle\zeta^i,\varphi\rangle_{\{Q\}}, \tag{3.21}$$

where the function φ_i is defined by (3.5).

Equations (3.18)–(3.21) can be used to calculate the derivatives

$$\frac{1}{N}\frac{\partial E_i}{\partial Q_i}=2\varepsilon_i(\lambda)\sigma_i(\lambda)\,|_{\lambda=Q_i}, \tag{3.22}$$

$$\frac{1}{N}\frac{\partial M_i}{\partial Q_j}=2\zeta_j{}^i(\lambda)\sigma_j(\lambda)\,|_{\lambda=Q_j}. \tag{3.23}$$

In both of these formulas we have used the parity properties of the functions ε_i, σ_i, ξ_j^i. The energy minimum is reached at $Q_i=Q_i^{(0)}$. According to (3.22), the limits $Q_i^{(0)}$ are determined by the system of equations

$$\varepsilon_i(\lambda;\{Q_j^{(0)}\})\,|_{\lambda=Q_i^{(0)}}=0,\quad i=1,\ldots,r. \tag{3.24}$$

Using the relations (3.18) and (3.21), we can calculate the matrix of second derivatives of the energy at the minimum point:

$$\frac{1}{N}\frac{\partial^2 E}{\partial Q_i \partial Q_j}\bigg|_{\{Q_k\}=\{Q_k^{(0)}\}} = 2\delta_{ij}\varepsilon_i'(\lambda)\sigma_i(\lambda)\big|_{\lambda=Q_i^{(0)}}. \quad (3.25)$$

Equations (3.23) and (3.25) show how the energy of states with a symmetrically filled Fermi band depends on the quantum numbers m_i (3.12).

The energy of excited states in a sector with specified quantum numbers can be calculated just as in Sec. 2 for the *XXZ* model. The result is

$$E=N\varepsilon_0+\frac{1}{N}\sum_{j=1}^{r}\frac{\varepsilon_j'(Q_j)}{\sigma_j(Q_j)}\left\{\frac{1}{4}[(\hat{\zeta}^{-1}\mathbf{m})_j]^2 + [(\hat{\zeta}\mathbf{k})_j]^2+d_j^{(+)}+d_j^{(-)}-\frac{1}{12}\right\}, \quad (3.26)$$

where

$$\varepsilon_0=\frac{1}{2\pi}\langle\varepsilon,\varphi'\rangle_{(Q)} \quad (3.27)$$

is the energy density of the ground state,

$$\hat{\zeta}_{ij}=\zeta_j^i(\lambda;\{Q_k^{(0)}\})\big|_{\lambda=Q_j^{(0)}} \quad (3.28)$$

and the numbers $d_j^{(\pm)}$ are given by (3.11). Finally, the term 1/12 in (3.26) gives the Euler-Maclaurin correction.

The ratios $\varepsilon_j'(\theta_j)/\sigma_j(Q_j)$ can be interpreted as the Fermi velocities in each band, $j=1,2,...,r$. The model possesses a conformally invariant continuous limit only if all these velocities are equal,

$$v_F=\frac{1}{2\pi}\frac{\varepsilon_j'(Q_j)}{\sigma(Q_j)}, \quad j=1,\dots,r. \quad (3.29)$$

If this condition holds, it follows from Eqs. (3.26), (3.10) that the dimensions for the state $\{m_i, k_i; J_p^{i(\pm)}, J_h^{i(\pm)}\}$ are equal to[7]

$$\Delta^{(\pm)}=\sum_{j=1}^{r}\left\{\frac{1}{2}\left(\hat{\zeta}\mathbf{k}\pm\frac{1}{2}\hat{\zeta}^{-1}m\right)_j^2+d_j^{(\pm)}\right\}. \quad (3.30)$$

As in the *XXZ* model, states with a completely filled Fermi band correspond to primary operators, while states with holes and particles within the Fermi band correspond to secondary representations. However, it turns out that there are many more secondary representations of a given dimension than is the case in the Virasoro algebra representation. This indicates that the symmetry group here is larger than the conformal group.

4. CONCLUSIONS

We will now describe the operator algebra for the Ashkin-Teller model, which is obtained for the bootstrap equations in Ref. 14 and has been described previously in Refs. 21, 22. The entire algebra is generated by four fundamental operators of dimension $\Delta^{(\pm)}=1/16$. Two of them correspond to the Ising order parameters $\sigma^{(1)}$, $\sigma^{(2)}$, the other two to the disorder parameters $\mu^{(1)}$ and $\mu^{(2)}$. The operator expansion of the Ising variables contains exponentials of the free boson field (1.8) which form a closed operator subalgebra. The subalgebra corresponding to the *XXZ* model consists of the operators $V_{n,m}$ (n even) which appear in operator expansions of the type $\sigma^{(1)}\sigma^{(1)}$ and $\sigma^{(1)}\mu^{(2)}$. This result has a natural phenomenological interpretation. Indeed, the *XXZ* model is related to the 6-vertex model of statistical physics by equations (3.1a), (3.1b). The latter model is a special case of the 8-vertex Baxter model[1], which in turn is a specialization of the Ashkin-Teller lattice model[1] for two interacting Ising sublattices. The edge variables in the 8-vertex model are expressed as products $\sigma^{(1)}\mu^{(2)}$ of the Ising variables at neighboring sites. The spin configuration in the 8-vertex model remains unchanged if the signs of $\sigma^{(1)}$ and $\mu^{(2)}$ are simultaneously reversed, and the same is true of the above operator expansions.

If $\zeta=1/\sqrt{2}$ ($\gamma=0$; $h=0$), the operators (1.8) and their secondary representations give a boson representation in the $SU(2)\times SU(2)$-invariant and conformally invariant Wess-Zumino theory, which has an anomaly in the $k=1$ current algebra (Refs. 36, 37). The conserved currents form a Kac-Moody algebra[13,36,37] and are expressible in terms of the following operators $V_{n,m}$:

$$J_\pm^{(+)}(z_+)=V_{\pm2,\pm1}, \quad J_\pm^{(-)}(z_-)=V_{\pm2,\mp1}, \quad J_s^{(\pm)}(z_\pm)=\frac{\partial}{\partial z_\pm}\varphi^{(\pm)}(z_\pm). \quad (4.1)$$

Here the subscripts $\pm$ denote the isotopic current indices, while the superscripts $\pm$ denote holomorphic and antiholomorphic components, respectively.

For magnets with a larger symmetry group in the isotropic case ($\gamma=0$), the operators (1.11) are similarly related to a boson representation in the $G\otimes G$-invariant Wess-Zumino theory.[36] The currents corresponding to the generators of a Cartan subalgebra can be expressed as

$$J_i^{(\pm)}(z_\pm)=\frac{\partial}{\partial z_\pm}\varphi_i^{(\pm)}(z_\pm) \quad (4.2)$$

in terms of the derivatives of the r-component boson field, while the currents corresponding to the roots α_i in a Cartan-Weyl basis of the Lie algebra $\mathcal{G}$ are expressible in terms of the operators (1.11):

$$J_{\alpha_i}^{(\pm)}(z_\pm)=\frac{\partial}{\partial z}V_{\alpha_i,\pm\Omega_i}(z_\pm), \quad \Omega_i=A_{ij}\omega_j, \quad A_{ij}=(\alpha_i,\alpha_j), \quad (4.3)$$

where $\mathbf{a}_i$ and ω_i are the simple roots and fundamental weights of $\mathcal{G}$.

The Wess-Zumino theory no longer describes magnets in the anisotropic case ($\gamma\neq0$). In the continuous limit, they should correspond to the generalization of the Ashkin-Teller model to the case of Z_{r+1} symmetry.

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1) Such as $CsCoCl_4$ (Ref. 2) and $CsNiF_3$ (Ref. 3), for example.
2) The pair correlation functions for these operators fall off as $(1/r)^{2\Delta}$ at large distances.
3) Here k is equal to the macroscopic momentum of the state divided by twice the Fermi momentum; m measures the deviation of the projection of the spin on the Z axis from its equilibrium value.

[4)]The dimensions for the states with $k = \pm 1, m = 0$ and $m = \pm 1, k = 0$, as well as the dimensions for the secondary representations $d^{(+)} = 1$, $d^{(-)} = 0$ and $d^{(+)} = 0, d^{(-)} = 1$ of the vacuum, where calculated in Ref. 8.

[5)]Also called the Yang-Baxter equations.

[6)]This generalization cannot be carried out uniquely for certain groups, since different R-matrices are related by an automorphism of the root system.[32,34] In what follows we will discuss only solutions corresponding to the trivial automorphism.

[7)]Condition (3.29) holds in the limit $Q_j = Q \to \infty$, and

$$(\hat{\xi}^2)_{ij} = (\alpha_i, \alpha_j)[2(1-\gamma/\pi)]^{-1/2},$$

the primary dimensions being given by Eq. (1.11).

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