Quasisimple waves in Korteweg-de Vries hydrodynamics

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The onset and development of the nondissipative shock wave (NSW) formed as the result of the "breaking" of a nonlinear wave is investigated by the method of modulation equations. The situation when the wave is passing through an unperturbed medium is studied, and in this case two Riemann variables are sufficient for the description of the NSW. Both a monotonically increasing and a nonmonotonic (localized) initial perturbation are considered. In both cases the general solution is found by linearizing the modulation equations by the hodograph method. The asymptotic behavior of the resulting solution at large x and t is investigated and these results are compared with the corresponding quasiclassical formulas of the method of the inverse scattering problem.

1. INTRODUCTION

In dispersive hydrodynamics the breaking of the wave front of a simple Riemann wave[1] leads to the appearance of a nondissipative shock wave (NSW)—a region filled with nonlinear fine-scale oscillations that expands continuously with time. The onset of the NSW and its structure in the case of weak nonlinearity, when the Korteweg-de Vries (KdV) equation is valid, have been investigated in Refs. 2 and 3 (see also Ref. 4). Here the use of the modulation equations of Whitham[5] proved to be very effective. An important property of these equations for dispersive KdV hydrodynamics is that they can be represented in a symmetric "Riemann" form. The number of Riemann variables here is equal to three. If the physical conditions of the problem make it possible to fix two of them, the modulaton equations have analytical solutions in the form of simple waves. It was solutions of this type, obtained in Ref. 2, that made it possible to study the NSW structure that arises in conditions of a sharp initial shock. At the same time, one can also distinguish a much more general class of nonlinear waves, when only one of the Riemann variables turns out to be fixed, and the generation and propagation of the waves are described by the variation of the other two.[6] Such waves are called quasisimple. In particular, any wave traveling through a stationary uniform medium is quasisimple. The present paper is devoted to the theory of quasisimple waves. The study of a quasisimple wave, as we shall see below, makes it possible to study the onset and development of the oscillatory structure of a NSW not only for a sharp discontinuity but also for a monotonically growing one, and also for a nonmonotonic initial perturbation. The latter case is especially interesting, since it permits one to study a spatially bounded (integrable) initial perturbation and thereby perform a direct comparison of the asymptotic solutions (of the KdV equations) obtained by two different methods—the method of modulated waves, and the method of the inverse scattering problem.

We shall formulate the problem and recall those properties of the modulation equations that we shall need. We consider the solution of the KdV equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\varepsilon^2\frac{\partial^3 u}{\partial x^3}=0, \qquad (1)$$

that satisfies the initial condition

$$u(x,\,0)=r_0(x)\geqslant 0,$$

where the dimensionless parameter $\varepsilon\ll 1$, and r_0 is a sufficiently slowly varying function of x: $r_0/|dr_0/dx|\gtrsim 1$. It is obvious that in Eq. (1) we can at first neglect the dispersion term $\varepsilon^2\partial^3u/\partial x^3$, so that $u(x,t)\approx r(x,t)$, where

$$r=r_0(x-rt). \qquad (2)$$

The important point is that if the function $r_0(x)$ describing the initial perturbation has a decreasing part, the front of the wave (2) will become steeper with time and, at a certain moment, the derivatives of the function $r(x,t)$ will become infinite (breaking of the front). Close to and after this time it is no longer possible to neglect the dispersion in Eq. (1).

It is well known that in the region of breaking the solution of Eq. (1) has an oscillatory character, and, more precisely, is a quasistationary wave.[2] The stationary wave for Eq. (1) has the form

$$u(\xi)=2b\,\mathrm{dn}^2[\varepsilon^{-1}(b/6)^{1/2}\xi,\ m]+U-{}^2/_3 b(2-m), \qquad (3)$$

where $\xi=x-Ut$, $m=s^2$ (s is the modulus of the Jacobi elliptic function dnξ), and U is the phase velocity of the wave; the parameters m, b, and U are arbitrary. The amplitude of the oscillations is $a=\frac{1}{2}(u_{\max}-u_{\min})=mb$. If the parameters m, b, and U of the wave (3) are assumed to be slowly varying functions of x and t (a quasistationary wave) their variation is described by the modulation equations of Refs. 2–4. Introducing in place of m, b, and U the Riemann variables r_j ($j=1,2,3$), with $r_1\leqslant r_2\leqslant r_3$, we can represent these equations in the very simple form

$$\frac{\partial r_j}{\partial t}+v_j(r)\frac{\partial r_j}{\partial x}=0,\qquad j=1,2,3. \qquad (4)$$

Here the "group velocities" are equal to

$$v_1=U-\frac{2}{3}b\frac{m}{1-\mu},\qquad v_2=U-\frac{2}{3}b\frac{mm_1}{\mu-m_1},$$
$$v_3=U+\frac{2}{3}b\frac{m_1}{\mu}$$

[$m_1=1-m$, and $\mu=E(m)/K(m)$, where K and E are complete elliptic integrals of the first and second kinds], and the original variables can be expressed in terms of the Riemann variables in the form

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$$b = r_3 - r_1, \quad m = (r_2 - r_1)/(r_3 - r_1), \quad U = {}^1/_3(r_1 + r_2 + r_3).$$

Oscillations whose parameters are described by Eqs. (4) occupy on the x axis the NSW region.[2,6] The left boundary of the NSW is the point $x = x^-(t)$, where $m = 0$ (the trailing edge), and the right boundary is the point $x = x^+(t)$, where $m = 1$ (the leading front). On these boundaries the solution of the system (4) should satisfy the conditions[1)] (Ref. 2)

$$r_3(x^-, t) = r(x^-, t), \tag{5}$$

$$r_1(x^+, t) = r(x^+ t). \tag{6}$$

The functions $x^-(t)$ and $x^+(t)$, together with the $r_j(x,t)$, are to be determined.

An important advantage of the Riemann form (4) is the possibility of considering the reduced systems that are obtained from (4) when one of the variables r_j takes a constant value; such solutions are called quasisimple waves. In the more general case it is possible to fix more than one variable r_j (Ref. 6). In the present paper we consider a wave propagating through a stationary gas, to which correspond

$$r_0(x) = 0 \quad \text{for} \quad x \geqslant 0, \quad r_0(x) > 0 \quad \text{for} \quad x < 0.$$

It follows then from the condition (6) on the leading front that $r_1 \equiv 0$, i.e., we are dealing with a quasisimple wave. For the variables r_2 and r_3 we obtain from (4) the system of equations

$$\frac{\partial r_j}{\partial t} + r_3 V_j(m) \frac{\partial r_j}{\partial x} = 0, \quad j = 2, 3 \quad (0 \leqslant r_2 \leqslant r_3), \tag{7}$$

where

$$m = \frac{r_2}{r_3}, \quad V_2 = \frac{1}{3}\left(m + 1 - \frac{2mm_1}{\mu - m_1}\right),$$

$$V_3 = \frac{1}{3}\left(m + 1 + \frac{2m_1}{\mu}\right),$$

supplemented by the condition (5) on the trailing edge.

We shall consider in more detail the initial condition on $r_0(x)$ for $x \leqslant 0$. In the present paper we have investigated two characteristic cases (see Fig. 1):

a) a monotonically decreasing function $r_0(x)$;

b) a function $r_0(x)$ with one maximum with $r_0(-\infty) = 0$ (a localized perturbation).

We assume that the derivative r_0' is a minimum on the leading front of the initial perturbation, i.e., at the point $x = 0$. Then the breaking of the front will occur, according to (2), precisely at this point. Therefore, with no loss of generality we can assume that $r_0'(-0) = -\infty$, so that the breaking occurs at the initial time $t = 0$ at the point $x = 0$. In order that there be no other breaking points, we shall assume that $r_0(x)$ is convex in the region of decrease (the presence of more than one breaking point will lead, generally speaking, to a more complicated, many-phase modulation theory).

The calculation is carried out for the degenerate case as well (case c in Fig. 1).

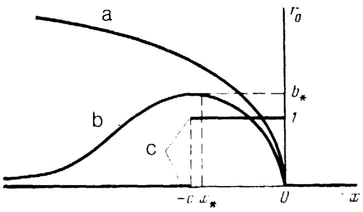

FIG. 1. Types of initial perturbation: a) monotonically increasing; b) nomonotonic (localized); c) a perturbation of constant magnitude, localized in a bounded region.

2. THE GENERAL SOLUTION

The quasilinear homogeneous system (7), consisting of two equations, can be linearized by means of a hodograph transformation. If we take r_2 and r_3 as the independent variables, and x and t as the functions to be determined, from (7) we obtain

$$\frac{\partial x}{\partial r_3} - r_3 V_2(m) \frac{\partial t}{\partial r_3} = 0, \quad \frac{\partial x}{\partial r_2} - r_3 V_3(m) \frac{\partial t}{\partial r_2} = 0. \tag{8}$$

The unknown boundaries $x = x^-(t)$ and $x = x^+(t)$ correspond in the hodograph plane to known boundaries $r_2 = 0$ and $r_2 = r_3$. The condition (5) takes the form

$$x(0, r_3) - r_3 t(0, r_3) = c(r_3), \tag{9}$$

where $x = c(r)$ is the inverse of the function $r = r_0(x)$.

Taking into account the form of the coefficients of the system (8), it is convenient to go over from the hodograph plane to the variables $m = r_2/r_3$ and $b = r_3$. The (8) and (9) are transformed into

$$\begin{aligned} &m \frac{\partial x}{\partial m} - mbV_2 \frac{\partial t}{\partial m} - b \frac{\partial x}{\partial b} + b^2 V_2 \frac{\partial t}{\partial b} = 0, \\ &\frac{\partial x}{\partial m} - bV_3 \frac{\partial t}{\partial m} = 0, \end{aligned} \tag{10}$$

$$x(0, b) - bt(0, b) = c(b). \tag{11}$$

The system (10), (11) admits a Mellin transformation, i.e., we can seek its solution in the form

$$\varphi(b) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} b^q \Phi(q) dq, \quad \Phi(q) = \int_0^\infty b^{-q-1} \varphi(b) db,$$

where $\varphi(b) = [x(m,b), bt(m,b)]$. The Mellin transformation is equivalent to the possibility of separating the variables and of finding a basis of solutions

$$x_q(m, b) = b^q X_q(m), \quad t_q(m, b) = b^{q-1} T_q(m), \tag{12}$$

where q are arbitrary numbers and the functions $X_q(m)$ and $T_q(m)$ satisfy the linear system

$$\begin{aligned} X_q' - V_2 T_q' &= m^{-1}[qX_q + (1-q)V_2 T_q], \\ X_q' - V_3 T_q' &= 0. \end{aligned} \tag{13}$$

We note that the solutions (12), when inverted to express m and b in terms of x and t, are generalized self-similar solutions of the scaling type:

$$m = f(xt^{-\alpha}), \quad b = t^\beta g(xt^{-\alpha}) \quad (\alpha = q/(q-1), \quad \beta = 1/(q-1)).$$

The system (13) reduces to a single first-order (Riccati) equation for the function $\zeta_q = X_q/T_q$:

$$\frac{d\zeta_q}{dm}+\frac{(\zeta_q-V_3)[q\zeta_q+(1-q)V_2]}{m(V_3-V_2)}=0. \qquad (14)$$

The function $T_q(m)$ is determined from $\zeta_q(m)$ by quadrature from the equation

$$\frac{dT_q}{dm}=\frac{q\zeta_q+(1-q)V_2}{m(V_3-V_2)}T_q, \qquad (15)$$

after which it is easy to find $X_q=\zeta_q T_q$. Typical phase portraits of Eq. (14) are given in Figs. 2a and 2b.

We note that the point $x^-(t)$ of the trailing edge of the quasisimple wave can be determined in explicit form before the problem of the variation of the modulation parameters inside the wave is completely solved. In fact, we introduce for the functions x and t on the boundary $m=0$ the notation $x(0,b)=x_0(b)$ and $t(0,b)=t_0(b)$. The equations $x=x_0(b)$ and $t=t_0(b)$ in parametric form determine the law of motion of the trailing front $x=x^-(t)$. From the first equation (10) for $m\to 0$ we obtain the relation

$$x_0'+bt_0'=0 \qquad (16)$$

[which is equivalent to the differential equation $dx^-/dx=-b(x^-,t)$]. The boundary $m=0$ is a characteristic curve of the system (10). The meaning of the relation (16) is that only when the boundary values of the functions x and t satisfy this condition does the system (10) have a solution.

From the boundary condition (10) it follows that

$$x_0'-bt_0'-t_0=c';$$

comparing this with the characteristic condition (16), we obtain the condition

$$2bt_0'+t_0+c'=0,$$

from which one can determine $t_0(b)$ by quadrature, and then, with the aid of (11), determine $x_0(b)$ as well:

$$t_0=b^{-1/2}F(b),\quad x_0=c(b)+b^{1/2}F(b), \qquad (17)$$

where

$$F(b)=-\frac{1}{2}\int b^{-1/2}c'(b)\,db. \qquad (18)$$

The question of the choice of the constant of integration in (18) is solved in a manner which depends on whether or not the function $r_0(x)$ is monotonic; in the monotonic case, obviously, $F(0)=0$. For a nonmonotonic function $r_0(x)$ there are two branches of the inverse function: $c^{\rm I}(b)$ and $c^{\rm II}(b)$ ($c^{\rm II}\leqslant c^{\rm I}$). The initial part of the curve $x=x^-(t)$ is determined by the formulas (17) and (18) with $c=c^{\rm I}$ and $F^{\rm I}(0)=0$, with the parameter b increasing from zero to a maximum value b_*. The subsequent behavior of the trailing edge is described by the same formulas with $c=c^{\rm II}$ and $F^{\rm II}(b_*)=F^{\rm I}(b_*)$, giving

$$F^{\rm II}(b)=F^{\rm I}(b_*)+\frac{1}{2}\int_b^{b_*} b_1^{-1/2}[c^{\rm II}(b_1)]'\,db_1;$$

here the parameter b decreases from b_* to zero.

3. A MONOTONIC INITIAL PERTURBATION

We shall assume that the function $r_0(x)$ decreases monotonically in the region $x<0$, i.e., the function $c(b)$ is single-valued. In view of the power-law character of the dependence of the basis solutions x_q and t_q (17) on b, it is necessary to represent the function $c(b)$ in the form of a combination of powers:

$$c(b)=\sum_q c_q b^q \quad (q>1). \qquad (19)$$

We note that, in the case under consideration, only terms with $q>1$ can appear in this sum (or series). The solution of the problem (10), (11), owing to the linearity, can be sought in the form of an expansion in the basis (12):

$$x=\sum_q \alpha_q b^q X_q(m),\quad t=\sum_q \alpha_q b^{q-1}T_q(m). \qquad (20)$$

In the preceding section we found (uniquely) for the functions $x(m,b)$ and $t(m,b)$ on the boundary $m=0$ values that are compatible with the boundary condition (11) and the characteristic condition (16). A solution of the system (10) with these given values on the boundary exists, but is not unique, because $m=0$ is a characteristic curve. In (20) there is a corresponding arbitrariness in the choice of the solutions of the system (13) for each q. In reality, however,

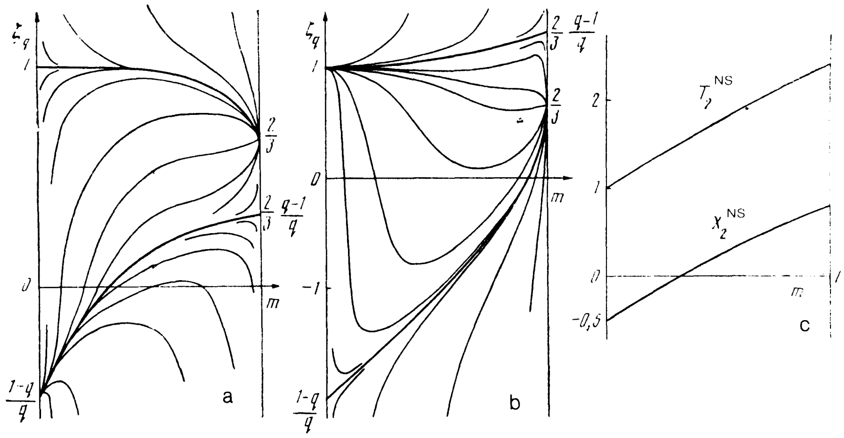

FIG. 2. Family of integral curves of Eq. (14): a) for the case $q>1$ (for the example of $q=2$); b) for the case $q<0$ (for the example of $q=-1$); c) bounded solution of the system (13) for $q=2$.

in the sum (20) each solution X_q, T_q should correspond to the separatrix (see Fib. 2a) that passes from the node at $m = 0$ to the saddle point at $m = 1$ (the NS separatrix). For all the other solutions, as can be shown, the domain in which the functions x and t in (20) are defined does not include the neighborhood of the boundary $m = 1$. Such solutions are not applicable to the description of the NSW, since joining with the outer solution $r(x,t)$ is possible only at the values $m = 0$ and $m = 1$ (Ref. 2).

We shall normalize the solution X_q^{NS}, T_q^{NS} corresponding to the NS separatrix by the condition

$$T_q^{NS}(0)=1 \tag{21}$$

(for $q > 1$ this is possible). Then $X_q^{NS}(0) = (1-q)/q$, and the condition (11), which, with allowance for (19) and (20), takes the form

$$\alpha_q[X_q^{NS}(0)-T_q^{NS}(0)]=c_q,$$

gives the possibility of finding the coefficients α_q. Finally, for the case of a monotonic initial perturbation we have

$$x(m,b)=\sum_q \frac{q}{1-2q} c_q b^q X_q^{NS}(m),$$
$$t(m,b)=\sum_q \frac{q}{1-2q} c_q b^{q-1} T_q^{NS}(m), \tag{22}$$

where the summation is performed over the same values of $q(>1)$ as in the expansion (19) of the function $c(b)$.

The solution X_q^{NS}, T_q^{NS} can be found numerically; the calculation is stable if one first integrates (14) from $m = 1$ $[\zeta_q(1) = (2/3)(q-1)q^{-1}]$, and then integrates (15) from $m = 0$ using (21).

We note that X_q^{NS}, T_q^{NS} is (apart from the normalization) the only solution of the system (13) that has a finite limit as $m \to 1$. Setting $m = 1$ in (22), we obtain a parametric representation of the motion of the leading front of the NSW, the velocity of which is $\dot{x}^+ = 2b(x^+, t)/3$.

As a simple example we shall consider the problem of a quasisimple wave developing from the initial profile

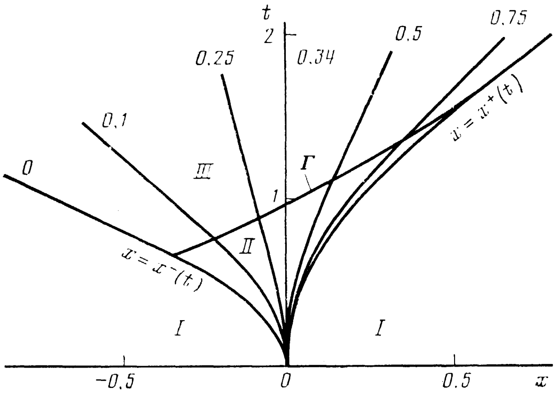

FIG. 3. Form of the NSW region in the x,t plane for the example of $r_0(x) = (-x)^{1/2}$ for $-1 < x < 0$, $r_0(x) = 1$ for $x < -1$. Lines of constant value of the function $m(x,t)$ are also shown.

$$r_0(x)=0 \text{ for } x\geqslant 0,\ r_0(x)=(-x)^{1/2} \text{ for } -1<x<0,$$
$$r_0(x)=1 \text{ for } x\leqslant -1.$$

The solution of this problem is described by different formulas in different regions of the x,t plane (see Fig. 3). In the region I external to the NSW it is determined by the relation (2). In region II the solution is given by the formulas (22), in which only the term with $q = 2$ ($c_2 = -1$) is kept. The law of motion of the trailing front is

$$x^- = -{}^3/_4 t^2 \text{ for } t\leqslant {}^2/_3,\ x^- = {}^1/_3 - t \text{ for } t > {}^2/_3.$$

The line Γ, which is the boundary between the regions II and III, is given by the equations $x = 2X_2^{NS}(m)/3$ and $2T_2^{NS}(m)/3$. Graphs of the functions $X_2^{NS}(m)$ and $T_2^{NS}(m)$ are illustrated in Fig. 2c. In region III the solution is a simple wave and cannot be found on the hodograph plane. In this wave $b \equiv 1$, and function $m = r_2$ satisfies the equation

$$\frac{\partial m}{\partial t} + V_2(m)\frac{\partial m}{\partial x} = 0,$$

the general solution of which has the form

$$m=M[x-V_2(m)t].$$

The function M for the given example is determined by a condition on Γ:

$$m=M[{}^2/_3 X_2^{NS}(m) - {}^2/_3 V_2(m) T_2^{NS}(m)].$$

It is obvious that m keeps a constant value on the straight-line characteristics

$$x-{}^2/_3 X_2^{NS}(m) = V_2(m)[t-{}^2/_3 T_2^{NS}(m)].$$

For large x and t the solution described above is transformed into a centered simple wave

$$x/t=V_2(m).$$

The expressions obtained now fully determine in implicit form the variation of the parameters $m = r_2/r_3$ and $b = r_3$ as functions of x and t. Using them in (3), we can construct a quasistationary wave that displays the oscillatory structure of the NSW. It has the usual form—it begins as a soliton with maximum amplitude and gradually degenerates into sinusoidal oscillations. In the example considered with $q = 2$ the amplitude of the head soliton near the breaking point grows with time: $a|_{m=1} = b^+ = 3\alpha t/2$, where $\alpha = [T_2^{NS}(1)]^{-1} \approx 0.42$. The velocity of the head soliton is $\dot{x}^+ = \alpha t$, and the trailing edge moves away into the region $x < 0$ with velocity $\dot{x}^- = -3t/2$, so that the region occupied by the NSW expands with time as

$$\Delta x = x^+ - x^- = {}^1/_2(\alpha + {}^3/_2)t^2 \approx 0.96 t^2.$$

The amplitude of the head soliton reaches its maximum value $b^+ = 1$ at $t = 2\alpha^{-1}/3 = 1.59$. After this the growth of the head soliton ceases. Asymptotically at large t the distance between neighboring solitons near the leading front increases logarithmically: $\lambda = 2\pi k^{-1} \approx \varepsilon\sqrt{6} \ln t$. The general pattern of the oscillations gradually approaches the pattern obtained in Ref. 2 in the solution of the problem of the decay of an initial discontinuity.

We note that in the case of cubic breaking considered in Ref. 2 the amplitude b^+ grew as $t^{1/2}$, and the region occupied by the NSW expanded as $t^{3/2}$.

4. A LOCALIZED INITIAL PERTURBATION

The aspect that is new in comparison with the monotonic case is that the function $c(b)$ is two-valued, and, as a consequence, the hodograph transformation $(x,t)\rightarrow(m,b)$ is two-sheeted. The independent variables m and b vary on two sheets; $0\leqslant m\leqslant 1, 0\leqslant b\leqslant b_*$ [$b_* = r_0(x_*)$ is the maximum value of $r_0(x)$]: Sheet I corresponds to the "nose" part of the initial perturbation (the part to the right of the maximum), while sheet II corresponds to the "tail" region. The characteristics on both sheets have the same equations: $b=\text{const}$ and $mb=\text{const}$ (corresponding to $r_j=\text{const}, j=2,3$), but the semicharacteristics, specifying the basins of influence, are different (see Fig. 4).

The plan for obtaining the desired solution is natural: First we find the solution on sheet I and then we find the solution on sheet II, using continuous splicing with the solution on sheet I at $b=b_*$. Because the system (10) is hyperbolic and the splicing line is a characteristic, this procedure is correct.[7] The formulas (12)–(15) used in the construction of the solution apply equally to both sheets, but the boundary-value problems are formulated differently. The branches of the function $c(b)$ are specified by distinct expressions on each sheet:

$$c^{\mathrm{I}}(b)=\sum_{q} c_q{}^{\mathrm{I}} b^q \qquad (q>1), \tag{23.I}$$

$$c^{\mathrm{II}}(b)=\sum_{q} c_q{}^{\mathrm{II}} b^q. \tag{23.II}$$

As a model, below we consider the particular case when the sums in (23) contain only one term each:

$$c^{\mathrm{I}}(b)=c_s b^s \ (s>1), \qquad c^{\mathrm{II}}(b)=c_{-p} b^{-p} \ (p>0).$$

The extension to the general case reduces to summing the solutions over q, and note is made of this at the appropriate points in the subsequent text. Without loss of generality, we set $b_*=1$; then the coefficients c_s and c_{-p} are the same, but it is convenient to retain different notation for them.

1. The solution on sheets I and II

The solution on sheet I is obtained in precisely the same way as in the monotonic case, and has the form

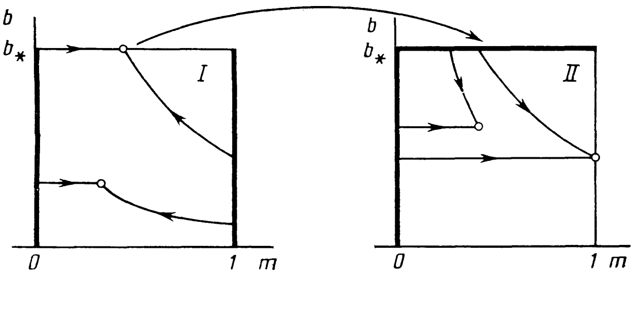

FIG. 4. Regions of dependence in the problems on sheets I and II for the system (10). The thick lines show those parts of the boundary from which the data are "carried over."

$$x^{\mathrm{I}}(m,b)=\frac{s}{1-2s} c_s b^s X_s{}^{\mathrm{yc}}(m),$$

$$t^{\mathrm{I}}(m,b)=\frac{s}{1-2s} c_s b^{s-1} T_s{}^{\mathrm{yc}}(m). \tag{24}$$

In the general case (23.I) it is necessary to sum (24) over s [cf. (22)].

The solution on sheet II requires a more detailed study of the system (13), the results of which are given in the Appendix. The conditions that single out the desired solutions are (11) with $c(b)=c^{\mathrm{II}}(b)=c_{-p}b^{-p}$, and

$$x^{\mathrm{II}}(m,1)=x^{\mathrm{I}}(m,1), \quad t^{\mathrm{II}}(m,1)=t^{\mathrm{I}}(m,1), \tag{25}$$

where the values of x^{I} and t^{I} for $b=1$ are taken from (24). In order to satisfy both these conditions it is necessary to use both the solutions $X_q^{(1)}$, $T_q^{(1)}$ and $X_q^{(2)}$, $T_q^{(2)}$ from the fundamental system (A1), (A2). The boundary condition (11) is satisfied by the following solution of the system (10):

$$x^{(2)}=-\frac{p}{1+2p} c_{-p} b^{-p} X_{-p}^{(2)}(m),$$

$$t^{(2)}=-\frac{p}{1+2p} c_{-p} b^{-p-1} T_{-p}^{(2)}(m) \tag{26}$$

[see (A3)–(A6)]. To (26) we must add the solution $x^{(1)}, t^{(1)}$ satisfying the homogeneous condition (11) [when $c(b)=0$], in order that for

$$x^{\mathrm{II}}=x^{(1)}+x^{(2)}, \quad t^{\mathrm{II}}=t^{(1)}+t^{(2)} \tag{27}$$

(25) be fulfilled. The general form of the solution satisfying the homogeneous condition (11) is as follows:

$$x^{(1)}=\sum_q \alpha_q^{(1)} b^q X_q^{(1)}(m),$$

$$t^{(1)}=\sum_q \alpha_q^{(1)} b^{q-1} T_q^{(1)}(m) \quad (q>-{}^3/_2)$$

[see (A5)]. The condition (25), with allowance for (24), (27), and (26), gives

$$\sum_q \alpha_q^{(1)} \begin{pmatrix} X_q^{(1)} \\ T_q^{(1)} \end{pmatrix} = \frac{s}{1-2s} c_s \begin{pmatrix} X_s^{\mathrm{yc}} \\ T_s^{\mathrm{yc}} \end{pmatrix} + \frac{p}{1+2p} c_{-p} \begin{pmatrix} X_{-p}^{(2)} \\ T_{-p}^{(2)} \end{pmatrix}. \tag{28}$$

Using (A7) and (A8) to represent the right-hand side here in the form of combinations of the solutions $(X_q^{(1)}, T_q^{(1)})$ (taking into account that $c_s=c_{-p}$), we find from (28) the coefficients $\alpha_q^{(1)}$. Finally, the solution on sheet II has the form

$$x^{\mathrm{II}}(m,b)=-\frac{p}{1+2p} c_{-p} b^{-p} X_{-p}^{(2)}(m) + \frac{s}{1-2s}\gamma_s c_s b^s X_s^{(1)}(m) + \sum_{k=0}^{\infty} \alpha_{k+1/2}^{(1)} b^{k+1/2} X_{k+1/2}^{(1)}(m),$$

$$t^{\mathrm{II}}(m,b)=-\frac{p}{1+2p} c_{-p} b^{-p-1} T_{-p}^{(2)}(m) + \frac{s}{1-2s}\gamma_s c_s b^{s-1} T_s^{(1)}(m) + \sum_{k=0}^{\infty} \alpha_{k+1/2}^{(1)} b^{k-1/2} T_{k+1/2}^{(1)}(m), \tag{29}$$

where

$$\alpha^{(1)}_{k+1/2}=\frac{s}{1-2s}c_s h_{ks}+\frac{p}{1+2p}c_{-p}h_{k,-p}, \tag{30}$$

and the coefficients γ_g and h_{kq} are defined in the Appendix (Secs. B and C). In the general case (23) it is necessary to sum (29) and (30) over those $s=q$ which appear in the sum (23.I), and over $p=-q$, where q appears in (23.II).

As an example we shall consider the solution (24), (29) in the case $s=2$, $p=1$. The corresponding solution of Eq. (1) at time $t=0.5$, when it has an already sufficiently developed NSW zone, is represented in Fig. 5; the corresponding picture in the x, t plane is depicted in Fig. 6b, in which the trailing and leading fronts and lines $b(x,t)=\text{const}$ are shown. Sheet I of the hodograph lane corresponds to region I, and sheet II corresponds to region II. The curve MN corresponds to the line of splicing of the sheets, and on it $b(x,t)$ is a maximum. At the time $t_M=2/3$ the tail of the rapidly oscillating wave reaches the maxium $r=1$ of the "outer" solution (2) (the point M on Fig. 6b), and the amplitude b^+ of the head soliton is equal at this time to 0.42 (the height of the soliton is equal to $2b^+$). At time $t_N=1.59$ the head soliton achieves its greatest amplitude: $b^+=1$ (the point N on Fig. 6b). After this it does not grow, and it moves with a constant velocity. The distance between the solitons near the leading front increases here linearly with time. The corresponding straight-line segment of the boundary of the NSW region in the x, t plane is carried over by the hodograph transformation to the point $m=1$, $b=1$ on sheet II. The other points of the right boundary of the sheet ($m=1$, $b<1$) are not reached at finite x and t—these are solitons of smaller amplitude that are realized only asymptotically (this asymptotic form is studied in detail below, in Sec. 2). Because of their low velocities they never move out to the leading front, on which the largest and fastest soliton is moving.

We shall consider an initial perturbation of constant magnitude $r_0=1$, concentrated in the interval from $-c$ to 0 (Fig. 1c). The outer solution on the trailing-edge side obviously has the form $r(x,t)=0$ for $x\leqslant -c$, $r(x,t)=(x+c)t^{-1}$ for $-c\leqslant x\leqslant t-c$, and $r(x,t)=1$ for $t-c\leqslant x\leqslant t$. The trailing edge moves in accordance with the law $x^-(t)=-t$ for $t\leqslant c/2$ and $x^-(t)=-c+c^2t^{-1}$ for $t\geqslant c/2$ [in accordance with Sec. 2, or directly from $dx^-/dt=-r(x^-,t), x^-(0)=0$]. That part of the initial perturbation where $r_0=1$ corresponds to a self-similar simple wave with $b(x,t)\equiv 1$ and $m(x,t)$:

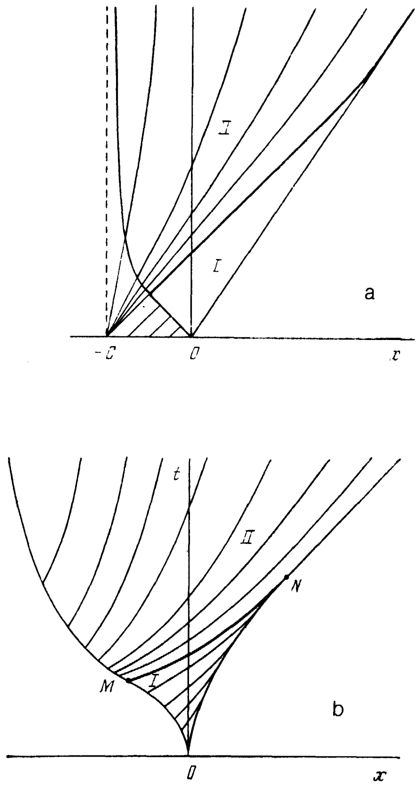

FIG. 6. Region of the NSW in the x,t plane (Fig. a corresponds to Fig. 1c, and Fig. b corresponds to Fig. 1b).

$$x/t=V_2(m) \tag{31}$$

(region I on Fig. 6a). The boundary of region I and region II, where $b<1$, is determined by the equation

$$dx/dt=V_3(m)$$

and by Eq. (31) with the initial condition $x=-c/2$, $t=c/2$ at $m=0$. It is found in the parametric form

$$t=t_*(m)=\frac{c}{2}\exp\int_0^m\frac{V_2'(m)\,dm}{V_3(m)-V_2(m)},$$

$$x=x_*(m)=V_2(m)t_*(m).$$

The quadrature here is calculated in explicit form in terms of $\mu=EK^{-1}$; we have ($m_1=1-m$)

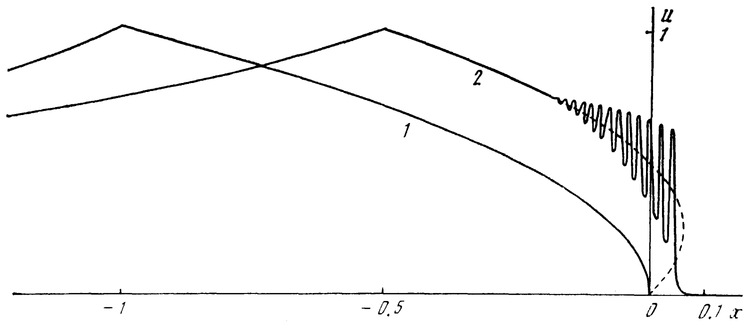

FIG. 5. Nondissipative shock wave (curve 2) that has developed (by the time $t=0.5$) from the initial perturbation given by $r_0(x)=0$ for $x\geqslant 0$, $r_0(x)=(-x)^{1/2}$ for $-1<x<0$, $r_0(x)=-x^{-1}$ for $x<-1$ (curve 1); $\varepsilon=2\cdot 10^{-3}$.

$$x_*(m)=\frac{1}{2}cm_1^{-1/2}\frac{(m+1)\mu-(3m+1)m_1}{(m+1)\mu-m_1},$$

$$t_*(m)=\frac{3}{2}cm_1^{-1/2}\frac{\mu-m_1}{(1+m)\mu-m_1}.$$

In region II the solution is found by the hodograph method. The boundary conditions (11) and (25) have in the given case the form

$$x(0,b)-bt(0,b)=-c, \tag{32}$$

$$x(m,1)=x_*(m),\ t(m,1)=t_*(m). \tag{33}$$

From (32) we find that $X_q^{(2)}$ and $T_q^{(2)}$ do not participate in the solution sought [see (A6)]. Using (33) and the expansion of t_* in powers of m,

$$t_*=c\Phi(m)=c\sum_{n=0}^{\infty}m^n\Phi_n,$$

we find the solution in region II in the hodograph representation:

$$x(m,b)=\sum_{k=0}^{\infty}\alpha_k b^{k+1/2}X_{k+1/2}^{(1)}(m),\qquad t(m,b)=\sum_{k=0}^{\infty}\alpha_k b^{k-1/2}T_{k+1/2}^{(1)},$$

where $\alpha_k=c\sigma_k$, and the σ_k are determined from (A12). The first coefficients α_k are $\alpha_0=c/2$, $\alpha_1=3c/16$, and $\alpha_3=15c/128$.

2. Approach of the solution to a soliton wave

We shall investigate the behavior of the solution (29) as $m\to 1$. The phase trajectories of all those solutions of the system (13) that participate in (29) approach the node $\zeta_q(1)=2/3$ as $m\to 1$. These solutions are not bounded as $m\to 1$; their asymptotic forms are

$$X_q(m)\approx {}^2/_3A_q[\Lambda+(q^{-1}-1)C_q]+(1-m)\{-{}^1/_3qA_q\Lambda+{}^1/_6A_q[2(q-1)C_q-1]\}, \tag{34}$$

$$T_q(m)\approx A_q(\Lambda-C_q)+(1-m)\{{}^1/_2(1-q)A_q\Lambda+{}^1/_4A_q[2(q-1)C_q-1]\}, \tag{35}$$

where $\Lambda=-\frac{1}{2}\ln(1-m)+\ln 4$, and A_q and C_q are certain constants that are uniquely determined for each solution X_q, T_q appearing in (29). It is clear that for a given b and $m\to 1$ the solutions $x=x^{\text{II}}(m,b)$ and $t=t^{\text{II}}(m,b)$ go off to infinity. If we substitute (34) and (35) into (29), neglecting corrections of order $(1-m)\Lambda$, after the elimination of Λ we obtain

$$x={}^2/_3bt+x_0(b), \tag{36}$$

where

$$x_0={}^2/_3\sum_q b^q\alpha_qA_qC_qq^{-1}$$

[α_q are the coefficients of the basis solutions (12) in (29)]. The relation (36) gives the asymptotic form of the function $b(x,t)$ for $x,t\to\infty$. It is a soliton wave—a chain of solitons with amplitude increasing in the direction of their motion (it follows that these are asymptotically noninteracting solitons). The soliton wave is a solution of the system of modulation equations (7) with $r_2=r_3$ (i.e., with $m=1$), when (7) degenerates into the single equation

$$\frac{\partial b}{\partial t}+\frac{2}{3}b\frac{\partial b}{\partial x}=0,$$

the general solution of which is given by the relation (36) with an arbitrary function $x_0(b)$. The straight lines (36) are lines of constant value of the function $b(x,t)$ for $x,t\to\infty$. To elucidate the character of the approach to the asymptotes it is necessary to take corrections to (34) and (35) into account. This gives

$$x=\frac{2}{3}bt+x_0(b)-\frac{16}{3}b[t+\theta(b)]\exp\left[-2\frac{t+\theta(b)}{\tau(b)}\right],$$

where

$$\tau=\sum_q b^{q-1}\alpha_qA_q,\qquad \theta=\sum_q b^{q-1}\alpha_qA_qC_q.$$

We shall estimate the behavior of m as $t\to\infty$ (for a given b) by confining ourselves in (35) to the leading term $T_q\approx A_q\Lambda$. Then

$$t=t^{\text{II}}\approx\Lambda(m)\tau(b), \tag{37}$$

whence $m\approx 1-16e^{-2t/\tau}$.

3. Comparison with the quasiclassical solution

We shall compare the soliton asymptotic form (studied in Sec. 2) of the solution in the case of a localized initial perturbation with the formulas obtained by Karpman[8] (see also Ref. 4, p. 598) by the method of the inverse scattering problem in the framework of the quasiclassical approximation: For $t\to\infty$ the coordinate of a soliton of amplitude b is equal to

$$x\approx{}^2/_3bt \tag{38}$$

and the wave number (soliton density) is

$$k(x,t)\approx\frac{1}{\varepsilon t}\varphi\left(\frac{3}{2}\frac{x}{t}\right)=\frac{1}{\varepsilon t}\varphi(b), \tag{39}$$

where

$$\varphi(b)=\frac{6^{1/2}}{8\pi}\int_{x_1(b)}^{x_2(b)}[r_0(x)-b]^{-1/2}dx$$

[$x_1<x_2$ are the roots of the equation $r_0(x)=b$]. Making here the replacement $r=r_0(x)$ we obtain for the quantity $\varphi(b)$ characterizing the soliton wave that the solution approaches as $t\to\infty$ a linear expression in terms of the difference $c^{\text{I}}-c^{\text{II}}$ describing the width of the initial perturbation:

$$\varphi(b)=\frac{6^{1/2}}{8\pi}\int_b^{b_*}(r-b)^{-1/2}\frac{d}{dr}[c^{\text{I}}(r)-c^{\text{II}}(r)]dr. \tag{40}$$

The linear transformation (40) is none other than an Abel transformation[9]

$$\psi(y)=\frac{1}{\pi}\int_0^y(y-y_1)^{-1/2}f'(y_1)dy_1,$$

relating the functions

$$f(y)=\frac{6^{1/2}}{8}[c^{\rm I}(b_*-y)-c^{\rm II}(b_*-y)],$$

$$\psi(y)=\varphi(b_*-y).$$

The inverse transformation

$$f(y)=\int_0^y (y-y_1)^{-1/2}\psi(y_1)\,dy_1$$

does not permit one to recover the initial perturbation $r_0(x)$ uniquely from $\varphi(b)$ (i.e., from the $t\to\infty$ asymptotic form)—only the difference $c^{\rm I}(b)-c^{\rm II}(b)$ is recovered. The reason is that we are dealing not with the exact formulas of the method of the inverse scattering problem, but wth the quasiclassical approximation of this problem. The degree of arbitrariness with which $r_0(x)$ is recovered in this approximation is in full agreement with the formulation indicated in the Introduction: Up to the breaking the evolution is described by Eq. (2), according to which each point of the wave profile moves with a constant velocity r, so that the difference $c^{\rm I}(r)-c^{\rm II}(r)$ does not change with time.

Comparison of the formulas (38) and (39) with (36) (in which for $t\to\infty$ we neglect x_0) and (37) with allowance for the expression for the wave number in terms of the parameters m and b

$$k=\frac{\pi}{\varepsilon K(m)}\left(\frac{b}{6}\right)^{1/2},\quad K(m)\approx\Lambda\quad\text{for}\quad m\to1,$$

leads to the relation

$$b^{1/2}\tau(b)=\frac{3}{4\pi^2}\int_b^{b_*}(r-b)^{-1/2}\frac{d}{dr}[c^{\rm I}(r)-c^{\rm II}(r)]dr. \qquad (41)$$

Analysis of this relation for the examples of the function $r_0(x)$ considered above in Sec. I shows that the expansions of the right- and left-hand sides of (41) contain the same powers of b, and comparison of the coefficients makes it possible to find the constants A_q in the leading terms of the expansions (34) and (35):

$$A^{(1)}_{k+1/2}=\frac{3}{4\pi^2}\frac{(2k-1)!!}{2^k k!}\sigma_k^{-1}\qquad(k=0,1,2,\dots),$$

$$A_q^{(2)}=-\gamma_q A_q^{(1)}=3(2\pi)^{-3/2}(1-2q)\frac{\Gamma(1/2-q)}{\Gamma(1-q)}\qquad(q\neq k+1/2).$$

Here the σ_k are determined from (A12).

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APPENDIX: ON THE SOLUTIONS OF THE SYSTEM OF EQUATIONS (18)

A. The fundamental system of solutions

Of greatest convenience for the use of the condition (11) is the system consisting of the solution

$$X_q^{(1)}=m^{q-1/2}\left[1+\frac{3}{8}m+\left(\frac{7}{32}-\frac{3}{32}\frac{2q-1}{2q+3}\right)m^2+\dots\right],$$

$$T_q^{(1)}=m^{q-1/2}\left[1+\frac{3}{8}m+\left(\frac{7}{32}+\frac{1}{32}\frac{2q-1}{2q+3}\right)m^2+\dots\right]\qquad(\mathrm{A1})$$

and the solution regular at the origin

$$X_q^{(2)}=\frac{1-q}{q}+\frac{3}{2}\frac{q-1}{2q-3}m+\frac{1}{16}\frac{(q-1)(38q-75)}{(2q-3)(2q-5)}m^2+\dots,$$

$$T_q^{(2)}=1+\frac{3}{2}\frac{q-1}{2q-3}m+\frac{1}{16}\frac{(q-1)(38q-75)}{(2q-3)(2q-5)}m^2+\dots\qquad(\mathrm{A2})$$

(the coefficients of the m^3 terms are already different). We remark that the fundamental system (A1), (A2) cannot be used for $q=1/2$, $\pm3/2$, $\pm5/2$,...: If $q=1/2$, (A1) and (A2) coincide, and for $q=\pm3/2$, $\pm5/2$,... one of them does not exist. For $q=k+\frac{1}{2}$ $(k\geqslant0)$ the solution (A1) is regular.

Substituting the solution of the system (10) of the general form (on each sheet)

$$x(m,b)=\sum_q b^q[\alpha_q^{(1)}X_q^{(1)}(m)+\alpha_q^{(2)}X_q^{(2)}(m)],$$

$$t(m,b)=\sum_q b^{q-1}[\alpha_q^{(1)}T_q^{(1)}(m)+\alpha_q^{(2)}T_q^{(2)}(m)]\qquad(\mathrm{A3})$$

into (11) and taking (23) into account, we obtain

$$\alpha_q^{(1)}[X_q^{(1)}(0)-T_q^{(1)}(0)]+\alpha_q^{(2)}[X_q^{(2)}(0)-T_q^{(2)}(0)]=c_q.\qquad(\mathrm{A4})$$

From (A1) and (A2) it follows that[2)]

$$X_q^{(2)}(0)-T_q^{(2)}(0)=(1-2q)/q,$$

$$X_q^{(1)}(0)-T_q^{(1)}(0)=0\quad\text{for}\quad q>-3/2.\qquad(\mathrm{A5})$$

Then from (A4) we find

$$\alpha_q^{(2)}=\frac{q}{1-2q}c_q,\qquad(\mathrm{A6})$$

and $\alpha_q^{(1)}$ remains arbitrary.

B. The solution bounded as $m\to1$

This solution, corresponding on the phase portraits (Fig. 2) to the NS separatrix, can be expanded in the fundamental system (A1), (A2):

$$\begin{pmatrix}X_q^{\rm NS}\\T_q^{\rm NS}\end{pmatrix}=\begin{pmatrix}X_q^{(2)}\\T_q^{(2)}\end{pmatrix}+\gamma_q\begin{pmatrix}X_q^{(1)}\\T_q^{(1)}\end{pmatrix},\qquad(\mathrm{A7})$$

where the coefficient γ_q is determined numerically by trigonal inversion from $m=0$ to $m=1$ and back. The unit coefficient in the first term agrees with the normalization (21).

C. Expansion of the regular solution in solutions with half-integer q

For $q=k+\frac{1}{2}$, $k\geqslant0$ the series (A1) contain integer powers m^ν with $\nu\geqslant k$. Because of this, we can re-expand the solution (A2) with $q\neq3/2$, $5/2$,... in the solutions (A1) with $q=1/2$, $3/2$, $5/1$, ... as follows:

$$X_q^{(2)} = \frac{1-2q}{q} + \sum_{k=0}^{\infty} h_{kq} X_{k+1/2}^{(1)},$$

$$T_q^{(2)} = \sum_{k=0}^{\infty} h_{kq} T_{k+1/2}^{(1)}. \qquad \text{(A8)}$$

The coefficients h_{kq} are calculated using the recursion formula

$$h_{kq} = T_{kq} - \sum_{\nu=0}^{k-1} S_{k\nu} h_{\nu q}, \qquad \text{(A9)}$$

where T_{nq} and S_{nk} are the coefficients of m^n in the expansions of $T_q^{(2)}(m)$ and $T_{k+\frac{1}{2}}^{(1)}(m)$. The latter, in their turn, are determined by the recursions

$$n>0: \; T_{nq} = \sum_{\nu=0}^{n-1} \Delta_{n\nu q} T_{\nu q}, \qquad T_{0q}=1, \qquad \text{(A10)}$$

$$n>k: \; S_{nk} = \sum_{\nu=k}^{n-1} \Delta_{n\nu,k+1/2} S_{\nu k}, \qquad S_{kk}=1, \qquad \text{(A11)}$$

where

$$\Delta_{n\nu q} = \frac{(\nu-q+1)\varkappa_{n-\nu} + \nu(qn^{-1}-1)\lambda_{n-\nu}}{2(n-q)+1},$$

in which $\varkappa_n$ and λ_n can be obtained from the expansions

$$V_2(m) = -1 + \sum_{n=1}^{\infty} \varkappa_n m^n, \qquad V_3(m) = 1 + \sum_{n=1}^{\infty} \lambda_n m^n.$$

Calculations using the formulas (A9)–(A11) give

$$h_{0q}=1, \qquad h_{1q} = \frac{3}{8}\frac{2q-1}{2q-3}, \qquad h_{2q} = \frac{15}{64}\frac{2q-1}{2q-5}.$$

In general it can be shown that

$$h_{kq} = \frac{2(2q-1)}{2q-(2k+1)} \sigma_k \qquad (k=0,1,2,\ldots),$$

where σ_k is determined from the system of equations

$$\sum_{k=0}^{n} S_{nk}\sigma_k = \Phi_n, \qquad n=0,1,2,\ldots, \qquad \text{(A12)}$$

in which ϕ_n is obtained from the expansion

$$\Phi_0 + m\Phi_1 + m^2\Phi_2 + \ldots = \frac{m_1^{1/2}}{\mu(V_3-V_2)} = \frac{3}{2} m_1^{1/2} \frac{\mu - m_1}{(1+m)\mu - m_1}.$$

[1] On the trailing edge the "outer solution" $r(x,t)$ is the largest of the three roots of Eq. (2), while on the leading front it is the smallest of the three.

[2] The solution $X_q^{(1)}$, $T_q^{(1)}$ for $q \leqslant -3/2$ is not used anywhere in the following.

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