Dynamics of Bloch oscillations in Josephson junctions

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The quantum motion of a particle with dissipation in a sloping periodic potential simulating the dynamics of Josephson junctions is considered. The idea of band motion of a particle executing Bloch oscillations is valid only for finite time scales, determined by the dissipation. At late times interference of the Bloch oscillations occurs. In the presence of an alternating current of frequency close to the Bloch-oscillation frequency ω_B there are no resonance singularities on the current-voltage characteristic.

I. INTRODUCTION

At the present time many authors are actively investigating dissipative quantum mechanics.[1–4] In large measure this interest has been stimulated by progress in the preparation of small-scale Josephson junctions, the dynamics of which can be described starting from ideas about a quantum-mechanical particle with friction. An important advance in the theory of these processes was made by Caldeira and Leggett,[1] who derived the effective action. Use of the latter has made it possible to make a systematic study of the interaction of a quantum particle with a thermostat.

As is well known, the properties of a Josephson junction are determined by the dynamics of a quantum particle moving in a sloping periodic potential, the slope of which is proportional to the given current across the junction. This potential, as a rule, is quasiclassical, and the ordinary Josephson effect corresponds to classical motion of a particle in this potential. Quantum effects, associated with the finite transmissivity of the barrier, have been confirmed experimentally[5] and are in agreement with the theory of Refs. 1–3. The phenomenon of the discreteness of the energy levels in the individual wells has also been confirmed in experiments on the stimulation of the decay of the current state by an electromagnetic field.[6] The data obtained are in good agreement with the theory of Refs. 7 and 8. Here, theoretical allowance for the normal electrons that lead to the dissipation effects yields a correct description of the observed pattern.

The adequacy of the description of the dynamics of Josephson junctions by means of the well developed apparatus of dissipative quantum mechanics has now been reliably established. This gives justification for considering other quantum-mechanical effects arising in the motion of a particle in a sloping periodic potential. One such effect is band motion of the particle in this potential, if the slope of the potential is sufficiently small that the repulsion of levels in neighboring wells outweighs the probability of tunneling between them. Of course, in this case, in order that the width of the band of allowed energies not be too small the potential should not be strongly quasiclassical. The problem of the band motion of a particle is of interest when the dissipation is not very great, so that the energy losses in the sub-barrier motion of the particle between neighboring minima are smaller than the distance in energy to the next band.

In Refs. 9 and 10 it was shown that, because of band motion, the current-voltage characteristic (CVC) of a Josephson junction in the limit of small current and dissipation possesses characteristic singularities (see also Ref. 11). In addition, it is extremely interesting to study the effect on the physics of Josephson junctions of Bloch oscillations of a particle in a sloping periodic potential in the presence of dissipation. In the absence of friction, the Bloch oscillations are undamped oscillations of the particle velocity with a frequency proportional to the slope of the potential.[12]

In the present paper, as in Refs. 9 and 10, we consider the dynamics of Josephson junctions on the basis of ideas about the band motion of a particle. The limit of a narrow band and weak friction is investigated. Under the action of an oscillatory field of frequency equal to a multiple of the Bloch frequency the resonance singularities on the static CVC are smoothed out in a finite time. This happens because the motion of the particles, leading to migration of the packet through the lowest band, is effected via the high-lying states. Particles are thrown into these states as a result of incoherent interaction with the thermostat.

In this paper we also find that the CVC of a Josephson junction in the low-current region under consideration has a substantial dependence on the magnitude of the dissipation.

2. QUANTUM INTERFERENCE OF BLOCH OSCILLATIONS

In the case of a Josephson junction the role of the coordinate is played by the phase difference 2φ across the junction. With neglect of dissipation the Lagrangian has the form

$$L_0=\frac{m}{2}\left(\frac{\partial\varphi}{\partial t}\right)^2-V(\varphi), \tag{1}$$

where the potential energy is determined by the relation

$$V(\varphi)=-E_J\cos 2\varphi-F\varphi. \tag{2}$$

Here the Josephson energy is $E_J=I_c/2e$ and $F=I/e$, where I is the current across the junction. The role of the mass is played by the quantity $m=C/e^2+3E_J/8\Delta^2$, where C is the capacitance of the junction and 2Δ is the magnitude of the gap in the spectrum of the superconductor.[3] The potential $V(\varphi)$ is depicted in Fig. 1. Owing to the small transmissivity of the potential barriers the ground-state level forms a narrow band of allowed energies in the periodic potential:

$$\varepsilon(k)=-\frac{\delta}{2}\cos\frac{\pi k}{e}-\sum_{n=2}^{\infty}\frac{\delta_n}{2}\cos\frac{\pi n k}{e}, \tag{3}$$

which is indicated by the solid line in Fig. 2. In the limiting case that we are considering, the width δ of the band is small in comparison with $\Omega_p=2(E_J/m)^{1/2}$, the distance in ener-

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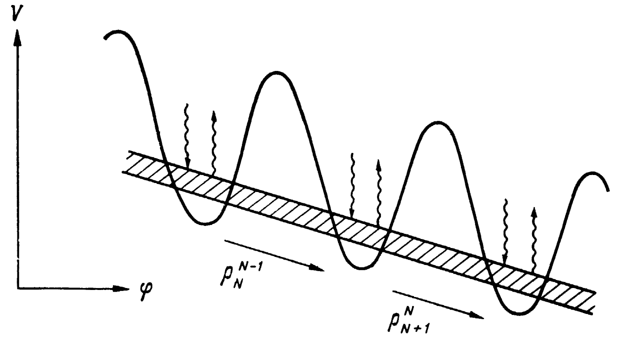

FIG. 1.

gy to the next level ($\Omega_p \ll E_J$). In this single-band approximation each succeeding term in Eq. (3) is smaller than the previous one by a factor proportional to the transmission coefficient of the barrier.

In the normalization chosen in Eq. (3) the quasimomentum $\tilde{k}$ may be called the quasicharge.[9] Umklapp processes then correspond to passage of a Cooper pair with charge $2e$ across the junction. Taking the quantum-mechanical expectation value of the Josephson relation $eV = \partial\varphi/\partial t$ gives for the voltage across the junction the formula $V = \partial\varepsilon(\tilde{k})/\partial\tilde{k}$. The voltage can take various values, depending on the position of $\tilde{k}$ in the Brillouin zone. For a sloping potential, in accordance with the quasiclassical dynamics of the particle, it is necessary to replace $\tilde{k}$ by $\tilde{k} + It$. When formula (3) is taken into account this yields the Bloch oscillations of the voltage:

$$V = \frac{\pi\delta}{2e}\sin\omega_B t + \frac{\pi\delta_2}{2e}\sin 2\omega_B t + \ldots \qquad (4)$$

Here we have introduced the Bloch-oscillation frequency $\omega_B = \pi I/e$. The damping of these oscillations in the absence of dissipation is proportional to the magnitude of the Zener breakdown and can be made very small in order not to degrade the oscillations over the period of the experiment.

The total direct current I across the junction is composed of the tunneling current between the edges of the junction and the charge-accumulation current $C\partial V/\partial t$ at the capacitor. Taking Eq. (4) into account we can obtain the time dependence of the charge $q(t)$ passing between the edges of the junction:

$$\frac{q(t)}{2e} = \frac{\omega_B}{2\pi}t - \frac{\pi\delta}{4}\frac{C}{e^2}\sin\omega_B t. \qquad (5)$$

In the narrow-band limit the second term gives rise to small smooth charge oscillations. It is important to learn how the Bloch oscillations are reflected in the properties of Josephson junctions in the presence of dissipation. Dissipation in a Josephson junction results either from shunting microshort-circuits along which ohmic current can flow or from impregnation of the superconducting edges of the junction by islands of the normal phase. Between these islands tunneling links occur, corresponding to the addition to the Langrangian (1) of fluctuation terms proportional to $\cos\varphi$ and $\sin\varphi$ (Refs. 2, 3). The corrections to the Lagrangian that correspond to shunting short circuits do not possess periodicity in the phase and, from which it follows that the quasimomentum ceases to be an exact quantum number.

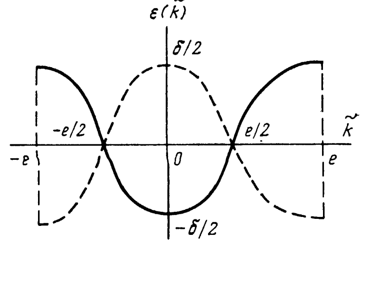

FIG. 2.

Terms of the $\cos\varphi$ type halve the Brillouin zone in Fig. 2, making it extend from $-e/2$ to $e/2$. The umklapp processes now correspond to a jump of a normal particle of charge e. Correspondingly, the spectrum will be represented by the solid and dashed lines in Fig. 2, and coherent motion of the particle through the original band from $-e$ to e, giving Bloch oscillations of frequency ω_B, becomes impossible. Nevertheless, coherent motion within the reduced band between $-e/2$ and $e/2$, corresponding to even n in formula (3), is not destroyed by a perturbation of the $\cos\varphi$ type. Accordingly, voltage oscillations with a frequency equal to an even multiple of ω_B become damped only when quantum tunneling effects are taken into account.[13] For such oscillations the harmonic of largest amplitude is given by the second term in Eq. (4).

Thus, in a real Josephson junction the voltage oscillations with a frequency equal to a multiple of ω_B damp in time with damping rate γ_1. However, this does not in itself imply that Bloch oscillations cannot affect the properties of a Josephson junction. In a system of a large number of monochromatic oscillators with frequencies continuously distributed about ω_B the ensemble-averaged bias damps in time with damping constant γ_1. Nevertheless, the spectral density of the noise, together with the response function describing the response to an oscillatory field of frequency ω_B, has a resonance character, with a width of the order of that of the distribution of the frequencies of the oscillators about ω_B. In our case, the Bloch oscillations of an individual particle in the band can serve as the analog of each such oscillator. In different language, an individual oscillator of frequency ω_B corresponds to a process of periodic accumulation of charge and of tunneling of a Cooper pair of charge $2e$ (Ref. 9).

In the present paper it is shown that resonance singularities of the noise spectral density in the steady state do not exist, being completely smoothed out over a time of order γ_2^{-1}. In the limit $T \gg \delta$ we have $\gamma_2 = \gamma_1(\delta/T)^2/24$. In the language of the system of oscillators this corresponds to a well defined interaction of the phases of their oscillations that is established in a time γ_2^{-1}. In this sense we can say that the reason for the absence of resonance effects is quantum interference of the Bloch oscillations, leading to their mutual extinction. The effect is a quantum one in the sense that the relaxation time γ_2^{-1} tends to infinity in the limit of high temperatures. In this case, the quasimomentum distribution function of the particles satisfies a differential equation of the Fokker-Planck type. At times $t \gtrsim \gamma_2^{-1}$ the quasimomentum description is not correct. As we shall see, the situation with the Bloch oscillations differs substantially from the ordinary time-dependent Josephson effect, when steady-state Shapiro steps are present.

In the following sections we shall analyze in more detail the general statements made here.

3. EQUATION FOR THE DENSITY MATRIX

The dynamics of a particle interacting with a thermostat can be described with the aid of a density matrix $\rho(\varphi, \tilde{\varphi}, t)$ (Refs. 14, 15), which depends on the time and two coordinates. Its value at the time t_f can be expressed in terms of its value at the time t_i by means of a functional integral:

$$\rho(\varphi, \tilde{\varphi}, t_f) = \int D\varphi D\tilde{\varphi} \exp\{iA[\varphi, \tilde{\varphi}]\}\rho(\varphi, \tilde{\varphi}, t_i). \quad (6)$$

The functional integral in Eq. (6) is taken over all values of $\varphi(t)$ and $\tilde{\varphi}(t)$ in the time interval between t_i and t_f. The action $A[\varphi, \tilde{\varphi}]$ is written in the form

$$A[\varphi, \tilde{\varphi}] = A_0[\varphi] - A_0[\tilde{\varphi}] + A_2[\varphi, \tilde{\varphi}], \quad (7)$$

where

$$A_0[\varphi] = \int_{t_i}^{t_f} dt\, L_0$$

is the action of the particle in the absence of dissipation.

The functional $A_2[\varphi, \tilde{\varphi}]$ describes the interaction of a quantum particle with the thermostat. In an ideal Josephson junction at zero temperature there are no dissipation mechanisms, since in superconductors in this case there are no normal excitations. In real junctions, however, as follows from experiment, there always exists a residual resistance. The reason for this may be the presence of impregnations of normal phase in the superconducting edges or the existence of shunting micro-short-circuits of normal metal. In the presence of impregnations of normal phase in the superconducting edges the quantity $A_2[\varphi, \tilde{\varphi}]$ is given by the expression

$$iA_2[\varphi, \tilde{\varphi}] = -\frac{\pi^2}{2}\alpha_T \int_{t_i}^{t_f} dt \int_{t_i}^{t} dt_1 \{K(t-t_1)\cos[\varphi(t)-\varphi(t_1)]$$

$$+ K(t_1-t)\cos[\tilde{\varphi}(t)-\tilde{\varphi}(t_1)]\}$$

$$+\frac{\pi^2}{2}\alpha_T \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt_1 K(t_1-t)\cos[\varphi(t)-\tilde{\varphi}(t_1)], \quad (8)$$

where $\alpha_T = 2/\pi e^2 R_T$, and the quantity R_T is the tunneling resistance between regions of normal phase. The function $K(t)$ is equal to

$$K(t) = \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} K(\varepsilon) e^{-i\varepsilon t}, \qquad K(\varepsilon) = \frac{\varepsilon}{\pi}\left(1 + \mathrm{cth}\frac{\varepsilon}{2T}\right). \quad (9)$$

Shunting short circuits of normal phase lead to dissipation that can be described by the same expression (8) with the substitutions

$$\alpha_T \to 4\alpha_s/\pi^2, \quad \alpha_s = \pi/2e^2 R_s, \quad (10)$$

where R_s is the resistance of the short circuits, and

$$\cos(\varphi-\varphi_1) \to -\tfrac{1}{2}(\varphi-\varphi_1)^2. \quad (11)$$

The expression (6) for the density matrix, together with Eqs. (7), (1), and (8) for the action, makes it possible to investigate the dynamics of a quantum particle with friction.

Below, we shall confine ouselves to treating the physically most interesting limiting case of a narrow allowed band with a width δ much smaller than the plasma frequency Ω_p. For the low-lying allowed bands the frequency Ω_p is equal to the spacing between the bands.

4. EQUATION FOR THE DENSITY MATRIX IN THE REPRESENTATION OF QUASILOCALIZED STATES

In the potential $V(\varphi)$ determined by Eq. (2) quasilocalized states appear (see Fig. 1). The lifetime of such states is long in proportion to the smallness of the Zener tunneling into the neighboring band. The presence of friction also reduces the lifetime of these states.

The wavefunctions of the quasilocalized states satisfy the equation[12]

$$\left[\varepsilon\left(-i\frac{\partial}{\partial\varphi}\right) - F\varphi\right]\Psi_N(\varphi) = \varepsilon_N \Psi_N(\varphi), \qquad \varepsilon_N = -\pi NF, \quad (12)$$

where $\varepsilon(k)$ is the spectrum of the lowest allowed band, given by Eq. (3); here, for convenience, we have changed to the quantity $k = \tilde{k}/e$, which varies between the limits -1 and $+1$. The wavefunctions $\Psi_N(\varphi)$ can be represented in the form[12]

$$\Psi_N(\varphi) = \sum_n w(\varphi - \pi n) \int_{-1}^{1} \frac{dk}{2}$$

$$\times \exp\left[i\pi k(n-N) - \frac{i}{F}\int_0^k dk_1 \varepsilon(k_1)\right], \quad (13)$$

where the Wannier functions $w(\varphi)$ are normalized to unity, and, in the narrow-band approximation,

$$w(\varphi) = (m\Omega_p/\pi)^{1/4} \exp(-m\Omega_p\varphi^2/2).$$

We expand the density matrix ρ in the eigenfunctions ψ_N

$$\rho(\varphi, \tilde{\varphi}, t) = \sum_{N,M} \rho_M{}^N(t)\Psi_N(\varphi)\Psi_M{}^*(\tilde{\varphi})\exp[i(\varepsilon_M - \varepsilon_N)t]. \quad (14)$$

As is well known, the magnitude V of the voltage across a tunnel junction is proportional to the rate of change of the phase φ. In our case the corresponding quantum-mechanical formula acquires the form

$$eV = \left\langle \frac{\partial\varphi}{\partial t} \right\rangle = \frac{\partial}{\partial t} \int d\varphi\, \varphi\rho(\varphi, \varphi, t), \quad (15)$$

where the angular brackets denote the quantum-mechanical expectation value. Below we shall neglect the dissipation mechanism associated with shunting short circuits.

To obtain the equation for the density matrix ρ we make use of the standard method of expanding the exponential in Eq. (6) in powers of $A_2[\varphi, \tilde{\varphi}]$. As a result, in first order in the quantity α_T we obtain[16]

$$\frac{\partial\rho_M{}^N}{\partial t} = \frac{\pi^2}{2}\alpha_T\left[-\rho_M{}^N \sum_{L=-\infty}^{\infty} K(-\varepsilon_L) J_L{}^2\left(\frac{\delta}{\pi F}\right)\right.$$

$$\left. + (-1)^{N+M} \sum_{L=-\infty}^{\infty} K(\varepsilon_L) J_L{}^2\left(\frac{\delta}{\pi F}\right)\rho_{M+L}^{N+L}\right]. \quad (16)$$

As is well known, in the expansion of the exponential in Eq. (6) in powers of $A_2[\varphi, \tilde{\varphi}]$, in first order in the interaction constant there appear not only the terms that lead to Eq.

(16) but also the so-called incoherent terms,[4] which, at sufficiently low temperatures, decay by a power law at times $\delta^{-1} \ll t_f - t_i \ll (\alpha_T \delta)^{-1}$ and exponentially at times $t_f - t_i \gg (\alpha_t \delta)^{-1}$ (Ref. 14). As well as the incoherent terms, terms of the from $\alpha_T [\exp(i\pi F t_f) - \exp(i\pi F t_i)]$, oscillating with the Bloch frequency ω_B, appear. The question arises as to how these oscillating terms behave at large times. To answer this question it is necessary to sum all the terms in the expansion of the right-hand side of Eq. (6) that are proportional to $\alpha_T^{n+1}(t_f - t_i)^n \exp(i\pi F t_{f,i})$.

We shall confine ourselves to investigating the simplest case of sufficiently high temperatures $T \gg \delta$. In this temperature range the incoherent terms damp with time as T^{-1}, and we shall not consider them below. The oscillating terms of interest are obtained by standard expansion of the right-hand side of Eq. (6) in powers of $A_2[\varphi, \tilde{\varphi}]$. As a result of rather long calculations, keeping only the terms proportional to $\alpha_T^n (t_f - t_i)^n$ and $\alpha_T^{n+1}(t_f - t_i)^n$ we obtain

$$\sum_N \rho^N_{N+2L+1}(t_f) = \sum_N \rho^N_{N+2L+1}(t_i) \exp[-\gamma_1(t_f - t_i)] + \frac{i\alpha_T \delta}{4F}$$
$$\times \exp[-\gamma_1(t_f - t_i)] \left[\sum_N \rho^N_{N+2L}(t_i) \exp(i\pi F t_i) - \sum_N \rho^N_{N+2L+2}(t_i) \right.$$
$$\left. \times \exp(-i\pi F t_i) \right] - \frac{i\alpha_T \delta}{4F} \exp[-\gamma_2(t_f - t_i)]$$
$$\times \left[\sum_N \rho^N_{N+2L}(t_i) \exp(i\pi F t_f) - \sum_N \rho^N_{N+2L+2}(t_i) \exp(-i\pi F t_f) \right]. \tag{17}$$

Equation (6) is reduced to a differential equation only when terms of the latter two types are neglected.

The relaxation rate γ_1 for arbitrary temperatures is determined from Eq. (16):

$$\gamma_1 = \pi^2 \alpha_T \sum_{N=-\infty}^{\infty} K(\varepsilon_N) J_N^2\left(\frac{\delta}{\pi F}\right). \tag{18}$$

In Eq. (17), corresponding to the high-temperature limit $T \gg \delta$, will have $\gamma_1 = 2\pi \alpha_T T$. In the same limit we obtain for the quantity γ_2 the expression

$$\gamma_2 = \alpha_T \pi \delta^2 / 12T. \tag{19}$$

Thus, the damping of the nondiagonal elements of the density matrix is determined by two times: γ_1^{-1} and γ_2^{-1}.

Equations (17) and (19) were obtained with the use of the high-temperature expansion of the kernel (9):

$$K(\varepsilon) = \pi^{-1}(2T + \varepsilon + \varepsilon^2/6T). \tag{20}$$

The relaxation rate γ_2 arises from the third term in formula (20).

Thus, the odd nondiagonal elements $\sum_N \rho^N_{N+2L+1}$ of the density matrix damp exponentially in time with damping times γ_1^{-1} and γ_2^{-1}. The damping of the even nondiagonal elements $\sum_N \rho^N_{N+2L}$ is due (in the absence of micro-short-circuits) to intraband-tunneling effects.[13]

To calculate the voltage across the junction we shall need the transition-matrix elements of the quantity φ between states N and M. Using Eqs. (3) and (13) we obtain

$$\int d\varphi \Psi_N(\varphi) \Psi_M^*(\varphi) \varphi = \pi N \delta_{N,M}$$
$$+ \frac{1}{F} \int_{-1}^{1} \frac{dk}{2} \varepsilon(k) \cos[\pi k(N-M)]$$
$$= \pi N \delta_{N,M} - \frac{1}{4F} \delta_{|N-M|}, \quad \delta_1 = \delta, \quad \delta_0 = 0. \tag{21}$$

From Eqs. (15) and (21) we find the rate of change of the coordinate φ:

$$\left\langle \frac{\partial \varphi}{\partial t} \right\rangle = \pi \frac{\partial}{\partial t} \sum_{N=-\infty}^{\infty} N \rho_N^N(t)$$
$$- \frac{1}{4F} \sum_{L=1}^{\infty} \delta_L \frac{\partial}{\partial t} \sum_N [\rho_N^{N+L} \exp(i\pi F L t) + \rho^N_{N+L} \exp(-i\pi F L t)]. \tag{22}$$

To first order in the parameter α_T, using Eq. (16) we obtain

$$\pi \frac{\partial}{\partial t} \sum_N N \rho_N^N(t) = -\frac{\pi^3}{2} \alpha_T \sum_N N K(\varepsilon_N) J_N^2\left(\frac{\delta}{\pi F}\right). \tag{23}$$

It follows from Eqs. (17) and (22) that in the absence of a shunt resistance and with neglect of interband transitions undamped oscillating components can appear in the voltage across the junction. Only the even harmonics of $eV(t)$, proportional to

$$\delta_{2N} \cos(2\pi N F t) \sim \delta (m\delta)^{2N-1} \cos(2\pi N F t)$$

can be weakly damped. In the narrow-band case that we are considering the effect is relatively small in the transmissivity of the barrier.

In the leading approximation we find from Eqs. (22) and (23) the value of the velocity[16]:

$$\langle \partial \varphi / \partial t \rangle = \pi \alpha_T \delta^2 / 4F, \tag{24}$$

i.e., in this approximation the velocity of the particle is independent of the temperature and inversely proportional to the magnitude of the slope of the periodic potential.

The approach developed in this section is based on perturbation theory in the parameter α_T. The spacing πF between neighboring levels of quasilocal states in this approach should exceed the width of these levels, which is proportional to the quantity γ_1. From this follows the region of applicability of Eq. (24): $\gamma_1 \ll F$. In addition, there exists a bound on the region of applicability of Eq. (24) on the side of high slopes F (high current). This is due to the fact that for large slopes of the potential the transmissivity of the barrier begins to depend on the quantity F. Allowing for this, the region of applicability of Eq. (24) is[16]

$$\alpha_T \max\{\delta; T\} \ll F \ll \Omega_p.$$

5. THE DENSITY MATRIX IN THE BAND APPROXIMATION

The above use of quasilocalized states made it possible to obtain results in the region of moderate values of the parameter F (the slope of the potential). In the region of small values of the parameter F such an approach becomes inconvenient because of the necessity of summing all the terms of the perturbation theory in the parameter γ_1/F.

To avoid these difficulties we shall make use of another method, based on the use of the quasimomentum representation. The spectrum of the lowest band is given by Eq. (3), and the Bloch wavefunctions, normalized to $2\pi\delta(k-k')$, are determined as

$$\Psi_k(\varphi)=\pi^{1/2}\sum_N w(\varphi-\pi N)\exp(i\pi kN). \tag{25}$$

The density matrix ρ in this representation takes the form

$$\rho(\varphi,\bar{\varphi},t)=\int_{-1}^{1}\frac{dk\,dk_1}{2\pi}\rho_{k_1}{}^{k}(t)\psi_k(\varphi)\psi_{k_1}^{*}(\bar{\varphi})\times\exp[it(\varepsilon(k_1)-\varepsilon(k))]. \tag{26}$$

Substituting the expression (26) for the density matrix into Eq. (15), for the magnitude of the velocity we obtain the expression

$$\left\langle\frac{\partial\varphi}{\partial t}\right\rangle=\int_{-1}^{1}dk\left\{\frac{\partial\varepsilon(k)}{\partial k}\rho_k{}^{k}(t)+\left[\frac{\partial\varepsilon(k)}{\partial k}\frac{\partial\rho_k{}^{k}}{\partial t}t+\frac{i}{2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial k}-\frac{\partial}{\partial k_1}\right)\rho_{k_1}{}^{k}\Big|_{k_1=k}\right]\right\}. \tag{27}$$

To obtain an equation for the density matrix $\rho_{k_1}^{k}$ we expand the quantity $\exp(iA)$ in Eq. (6) in powers of A_2 and $F\int\varphi dt$. As a result, in first order in both these parameters we obtain[12]

$$\frac{\rho_{k_1}{}^{k}(t_f)-\rho_{k_1}{}^{k}(t_i)}{t_f-t_i}=\frac{\pi^2\alpha_T}{2}\left\{i\rho_{k_1}{}^{k}(t_i)\int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi}K(\varepsilon)\times\left[\frac{1}{2\varepsilon(k_1)-\varepsilon-i\nu}-\frac{1}{2\varepsilon(k)-\varepsilon+i\nu}\right]+\rho_{k_1-\operatorname{sgn}k_1}^{k-\operatorname{sgn}k}K(-2\varepsilon(k))\times\frac{\exp[2it_f(\varepsilon(k)-\varepsilon(k_1))]-\exp[2it_i(\varepsilon(k)-\varepsilon(k_1))]}{2i(t_f-t_i)[\varepsilon(k)-\varepsilon(k_1)]}\right\}-F\left(\frac{\partial}{\partial k}+\frac{\partial}{\partial k_1}\right)\rho_{k_1}{}^{k}(t_i)+\frac{iF}{2}(t_f+t_i)\rho_{k_1}{}^{k}(t_i)\left[\frac{\partial\varepsilon(k)}{\partial k}-\frac{\partial\varepsilon(k_1)}{\partial k_1}\right]. \tag{28}$$

In the absence of a nonlocal part of the action A_2, according to Feynman and Hibbs,[15] in the derivation of the equation for the density matrix we can take the initial time t_i and final time t_f to be infinitesimally close. Then we obtain an exact differential equation for the density matrix. However, because of the nonlocality of the action A_2, in the quasimomentum representation, as in the representation of quasilocalized states, it is not possible, in the general case, to derive a closed differential equation. Inasmuch as in Eq. (28) we have not taken into account relaxation processes occurring with rate γ_2, the equation obtained from it for the diagonal part of the density matrix[10,16]:

$$\left(\frac{\partial}{\partial t}+F\frac{\partial}{\partial k}\right)\rho_k{}^{k}(t)=\frac{\pi^2\alpha_T}{2}[\rho_{k-\operatorname{sgn}k}^{k-\operatorname{sgn}k}K(-2\varepsilon(k))-\rho_k{}^{k}K(2\varepsilon(k))], \tag{29}$$

is valid only for a limited time interval $t\ll\gamma_2^{-1}$.

At high temperatures $T\gg\delta$ the relaxation rate $\gamma_2\ll\gamma_1$ and Eq. (19) can be used to calculate the static part of the CVC. At lower temperatures this equation gives the correct results in two cases: large potential slope $F\gg\alpha_T\delta$, and small slope $F\ll\alpha_T\delta$.

It follows from Eq. (28) that it is possible to write a closed equation for the diagonal part of the density matrix.[10,16]

In the limit $F\to 0$ the relaxation rate γ_2 also tends to zero, while the rate of the relaxation processes determined by Eq. (29) remains finite, which makes it possible to use Eq. (29) to find the CVC. For $F\gg\alpha_T\delta$, Eq. (29), as will be shown below, permits one to obtain the correct answer (formula (24) for the CVC.

According to Eq. (27), to find the velocity of the particle it is necessary to know also the nondiagonal part of the density matrix. The equation for this part can also be obtained from Eq. (28), and as a result the expression (27) for the velocity takes the form

$$\left\langle\frac{\partial\varphi}{\partial t}\right\rangle=\int_{-1}^{1}dk\frac{\partial\tilde{\varepsilon}(k)}{\partial k}\rho_k{}^{k}, \tag{30}$$

where the renormalized spectrum $\tilde{\varepsilon}(k)$ is determined by the expression

$$\tilde{\varepsilon}(k)=\varepsilon(k)\exp\left[-\pi^2\alpha_T\int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi}\frac{K(\varepsilon)}{\varepsilon^2-4\varepsilon^2(k)}\right]. \tag{31}$$

Equation (28) permits one to obtain in the expression (30) only the term of first order in α_T in the expansion of the exponential in (31).

Thus, the term linear in t in the expression (28) for the velocity has been cancelled, and, with logarithmic accuracy, Eq. (30) coincides with the usual expression for the current, in which the renormalized expression for the spectrum now appears.

Henceforth it is convenient to go over to the new functions

$$W_k(t)=\rho_k{}^{k}(t)+\rho_{k-\operatorname{sgn}k}^{k-\operatorname{sgn}k}(t),\qquad Y_k(t)=\rho_k{}^{k}(t)-\rho_{k-\operatorname{sgn}k}^{k-\operatorname{sgn}k}(t). \tag{32}$$

It follows from Eq. (29) that $W_k(t)$ is an arbitrary function of the variable $k-Ft$:

$$W_k(t)=W(k-Ft). \tag{33}$$

This property of the function W is destroyed by intraband-tunneling quasimomentum-umklapp process.[13] Taking the terms proportional to α_T^2, $\alpha_T F$, and F^2 into account gives the expression for $Y_k(t)$ given in the Appendix.

We shall express the density matrix $\rho_{k_1}^{k}(t)$ in terms of the matrix ρ_M^N. Comparing the two representations for the density matrix (Eqs. (14) and (26)), we find

$$\rho_{k_1}{}^{k}(t)=\sum_{N,M}\rho_M{}^{N}(t)\times\exp\left[i\pi F(N-M)t+i\pi k_1M-i\pi kN-\frac{i}{F}\int_{k_1}^{k}dk'\varepsilon(k')\right]. \tag{34}$$

From this we obtain an expression for the function $Y_k(t)$:

$$Y_k(t)=2\sum_{L=-\infty}^{\infty}\exp[i\pi(2L+1)(k-Ft)]\left\{\sum_N \rho_{N+2L+1}^N(t)\right\}. \quad (35)$$

As was shown above, the quantities in the curly brackets in Eq. (35) damp exponentially in time with damping constants γ_1 and γ_2, and this demonstrates once more the inapplicability of the quasimomentum approximation (Eq. (29)) at large times.

6. DEPENDENCE OF THE VELOCITY ON THE SLOPE OF THE POTENTIAL

As we noted above, to find the static part of the CVC it is possible to make use of expression (30) in the quasimomentum representation. In this expression the diagonal part of the density matrix satisfies the equation

$$-F\frac{\partial Y_k}{\partial k}=\frac{\pi^2\alpha_T}{2}\times\{Y_k[K(2\tilde{\varepsilon}(k))+K(-2\tilde{\varepsilon}(k))]+K(2\tilde{\varepsilon}(k))-K(-2\tilde{\varepsilon}(k))\}, \quad (36)$$

given in the Appendix. Solving Eq. (36) and substituting its solution into Eq. (30), we obtain

$$\left\langle\frac{\partial\varphi}{\partial t}\right\rangle=\frac{\pi^2\alpha_T}{4F}\left\{\frac{2}{1+\mathscr{F}(1)}\int_0^1 dk_1\frac{\partial\tilde{\varepsilon}(k_1)}{\partial k_1}\mathscr{F}(k_1)\times\int_0^1 dk\mathscr{F}(k)[K(-2\tilde{\varepsilon}(k))-K(2\tilde{\varepsilon}(k))]+\int_0^1 dk\int_0^k dk_1\frac{\mathscr{F}(k)}{\mathscr{F}(k_1)}\left[\frac{\partial\tilde{\varepsilon}(k)}{\partial k}(K(-2\tilde{\varepsilon}(k_1))-K(2\tilde{\varepsilon}(k_1)))-\frac{\partial\tilde{\varepsilon}(k_1)}{\partial k_1}(K(-2\tilde{\varepsilon}(k))-K(2\tilde{\varepsilon}(k)))\right]\right\}, \quad (37)$$

where the function $\mathscr{F}(k)$ is defined by the formula

$$\mathscr{F}(k)=\exp\left\{-\frac{\pi^2\alpha_T}{2F}\int_0^k dk_1[K(2\tilde{\varepsilon}(k_1))+K(-2\tilde{\varepsilon}(k_1)]\right\}. \quad (38)$$

The expression (37) differs from the corresponding result of Ref. 16 only in that Eq. (37) contains the renormalized spectrum.

At zero temperature and small values of the slope F, values of k and k_1 lying in a narrow neighborhood of the point $\frac{1}{2}$ are important. The integral J determining the renormalization of the spectrum, equal to

$$J=\int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi}\frac{K(\varepsilon)}{\varepsilon^2-4\varepsilon^2(k)}, \quad (39)$$

diverges logarithmically at large values of the energy ε and must be cut off by a value $\varepsilon=\omega_c\sim\Omega_p$. For $T=0$,

$$J=\int_0^{\omega_c}\frac{\varepsilon\, d\varepsilon}{\varepsilon^2-4\varepsilon^2(k)}=\frac{1}{\pi^2}\ln\left|\frac{\omega_c}{\delta\cos(\pi k)}\right|. \quad (40)$$

From Eqs. (31) and (40) we find the renormalized value of the spectrum:

$$\tilde{\varepsilon}(k)=-{}^1/_2\delta\cos(\pi k)\exp(-\alpha_T\ln|\omega_c/\delta\cos(\pi k)|). \quad (41)$$

Substituting the value found for the spectrum $\tilde{\varepsilon}(k)$ into Eq. (37) and taking into account that the quasimomenta k and k_1 are close to the value $\frac{1}{2}$, we obtain

$$\left\langle\frac{\partial\varphi}{\partial k}\right\rangle=\frac{\pi^3\alpha_T\delta^2}{4F}\left(\frac{\pi\delta}{\omega_c}\right)^{2\alpha_T}(1+\alpha_T)\int_{-\infty}^{\infty}dz\int_{-\infty}^{z}dz_1(z-z_1)|zz_1|^{\alpha_T}\times\exp\left[-\frac{\pi^2\alpha_T\delta}{2F(1+\alpha_T/2)}\left(\frac{\pi\delta}{\omega_c}\right)^{\alpha_T}(z|z|^{1+\alpha_T}-z_1|z_1|^{1+\alpha_T})\right]. \quad (42)$$

Calculating the integrals appearing in this formula, we find the value of the velocity:

$$\left\langle\frac{\partial\varphi}{\partial t}\right\rangle=\delta\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{\delta}{\omega_c}\right)^{\alpha_T/2}\left(\frac{F}{\alpha_T\delta}\right)^{1/2+\alpha_T/4}\times\left[1+\frac{\alpha_T}{4}(3+\pi-\ln 2-C)\right], \quad (43)$$

where $C=0.577$ is the Euler constant.

Thus, allowance for the terms of the next orders in α_T leads to a change of the dependence of the velocity on the magnitude of the slope F and to replacement of the bandwidth δ by its renormalized value $\tilde{\delta}$:

$$\tilde{\delta}=\delta(\delta/\omega_c)^{\alpha_T}. \quad (44)$$

For large values of the slope of the potential, in the region $\alpha_T\tilde{\delta}\coth(\tilde{\delta}/T)\ll F$, the velocity can be found from Eq. (36) by perturbation theory:

$$\frac{\partial Y_k}{\partial k}=-\frac{2\pi\alpha_T}{F}\int_0^k dk_1\tilde{\varepsilon}(k_1),$$

$$\left\langle\frac{\partial\varphi}{\partial t}\right\rangle=\frac{\pi^2\alpha_T\delta^2}{4F}\left(\frac{\delta}{\omega_c}\right)^{2\alpha_T}\left[1+\alpha_T\left(1-\frac{2\ln 2}{\pi}\right)\right], \quad (45)$$

With logarithmic accuracy, allowance for the terms of the next higher orders in α_T leads to the replacement of the bandwidth δ by its renormalized value $\tilde{\delta}$.

7. EFFECT OF THE DISSIPATION MECHANISM DUE TO SHUNTING SHORT CIRCUITS ON THE DYNAMICS

The presence of short circuits of normal metal shunting the Josephson junction can be taken into account using the action A_2 [Eq. (8)], in which one must make the substitutions (10) and (11). When only this dissipation mechanism is present, in the representation of quasilocalized states the equations for the density matrix have the form

$$\frac{\partial\rho_M{}^N}{\partial t}=\frac{\alpha_s\delta^2}{8F^2}\{\rho_{M+1}^{N+1}K(-\pi F)+\rho_{M-1}^{N-1}(\pi F)-\rho_M{}^N[K(-\pi F)+K(\pi F)]\}-2\pi\alpha_s T(N-M)^2\rho_M{}^N. \quad (46)$$

Summing over N in the right- and left-hand sides of this equation with a fixed difference $N-M$, we obtain

$$\left(\frac{\partial}{\partial t}+2\pi\alpha_s TL^2\right)\sum_N\rho_N^{N+L}=0. \quad (47)$$

The quantities $\sum_N\rho_N^{N+L}$, determined by Eq. (47), damp exponentially in time, and this leads, according to Eq. (20), to damping of the oscillatory part of the voltage. From Eqs. (20) and (46) we find the velocity:

$$\langle \partial\varphi/\partial t\rangle = \pi\alpha_s\delta^2/4F. \quad (48)$$

From comparison of Eqs. (22) and (48) it can be seen that in the region of large slopes of the potential the two dissipation mechanisms give the same dependence of the velocity on F. The same result was obtained in Refs. 9 and 10. For the dissipation mechanism considered here both the odd and the even elements of the density matrix damp in time.

8. EFFECT OF A HIGH-FREQUENCY FIELD ON THE DYNAMICS OF A QUANTUM PARTICLE

The action of a high-frequency field on a quantum particle can be described by means of an extra term

$$-F_1\varphi\cos\omega t \quad (49)$$

in the potential (2), where $F_1 = I_1/e$, I_1 being the amplitude of the alternating current.

Expanding the right-hand side of Eq. (6) in I_1 as well, in analogy with Eq. (16) we obtain

$$\frac{\partial\rho_M{}^N}{\partial t} = \frac{\pi^2}{2}\alpha_T\Big\{-\rho_M{}^N\sum_{L=-\infty}^{\infty}K(-\varepsilon_L)J_L{}^2\Big(\frac{\delta}{\pi F}\Big) + (-1)^{N+M}\sum_{L=-\infty}^{\infty}K(\varepsilon_L)J_L{}^2\Big(\frac{\delta}{\pi F}\Big)\rho_{L+M}^{L+N}\Big\}$$
$$-\frac{i\delta}{8}\frac{F_1}{F}\{(\rho_M^{N+1}-\rho_{M-1}^{N})\exp[i(\pi F-\omega)t] + (\rho_M^{N-1}-\rho_{M+1}^{N})\exp[i(\omega-\pi F)t]\}. \quad (50)$$

The last term in Eq. (50) describes the resonance effect of a high-frequency field on the dynamics of a quantum particle. In a potential with a floor this term leads to a resonance at frequencies ω close to πF. In our case, however, the summation between infinite limits leads to the disappearance of this effect.[16]

In the same way that we obtained Eq. (17) as an extension of Eq. (16), we can obtain the extension of Eq. (50). Without taking into account the relaxation occurring with rate γ_2, we obtain

$$\Big[\frac{\partial}{\partial t}+2\pi\alpha_T\frac{1-(-1)^L}{2}+i\pi LF_1\cos\omega t\Big]\sum_N\rho_{N+L}^N = \frac{\pi\alpha_T\delta}{4}\frac{1-(-1)^L}{2}\sum_N[\rho_{N+L-1}^N\exp(i\pi Ft) + \rho_{N+L+1}^N\exp(-i\pi Ft)]. \quad (51)$$

Equation (51) is valid in the approximation $T\gg\delta$ and over times $t\ll\gamma_2^{-1}$. The undamped right-hand side in Eq. (51) leads to a resonance effect of the oscillatory field on the CVC, and also to a resonance in the spectral density of the correlation function of the voltage.[10] However, over times $t\gtrsim\gamma_2^{-1}$ all these effects vanish exponentially with time because of the damping of the right-hand side of Eq. (51).

We note that this phenomenon is essentially related to the infinite extent of the periodic part of the potential energy (2). This does not occur in problems pertaining to a Stark ladder (Fig. 1) in real crystals,[17] near the boundaries of which the present approach is inapplicable.

9. CONCLUSION

We have investigated the dynamics of a quantum particle interacting with a thermostat. We have found that the equation of motion (29) for the density matrix, obtained in first order in the parameters F and α_T, is valid only over a restricted time interval. This equation gives an incorrect description of the behavior of the system at large times, since it does not contain the relaxation rate γ_2. At large times $t\gtrsim\gamma_2^{-1}$ the nondiagonal elements of the density matrix damp exponentially in time, and this leads to the disappearance of the resonance singularities on the CVC.

In a time γ_2^{-1} restructuring of the kinetics of the system occurs. The flux of particles through the lowest allowed band (the first term in the right-hand side of Eq. (27)) vanishes at times $t>\gamma_2^{-1}$. The total particle flux involves processes of transition to higher-lying states, represented by the wavy lines in Fig. 1. These processes occur on account of incoherent interaction with the thermostat and ensure the effective motion of the particle density through the lowest band.

APPENDIX

When $\exp\{iA[\varphi,\tilde\varphi]\}$ in Eq. (6) is expanded in powers of A_2 and F, terms of two types arise: incoherent terms that decay by a power law over times $t_f - t_i \ll (\max\{\alpha_T\delta;T\})^{-1}$ and decay exponentially at large times, and terms regular in $t_f - t_i$. Some of the latter reduce to a renormalization of the spectrum $\varepsilon(k)$. Omitting the incoherent terms and performing the renormalization of the spectrum, after rather lengthy calculations to second order in the parameters α_T and F we obtain, besides Eq. (32) for the function $W_k(t)$, an equation for the function $Y_k(t)$:

$$Y_k(t_f)=Y_k(t_i)-F(t_f-t_i)\frac{\partial Y_k(t_i)}{\partial k} + \frac{F^2}{2}(t_f-t_i)^2\frac{\partial^2Y_k(t_i)}{\partial k^2} - \frac{\pi^2\alpha_T}{2}(t_f-t_i)\cdot$$
$$\times\{Y_k(t_i)[K(2\bar\varepsilon(k))+K(-2\bar\varepsilon(k))] + W_k(t_i)[K(2\bar\varepsilon(k))-K(-2\bar\varepsilon(k))]\}$$
$$+\Big(\frac{\pi^2\alpha_T}{2}\Big)^2\frac{(t_f-t_i)^2}{2}\{Y_k(t_i)[K(2\bar\varepsilon(k))+K(-2\bar\varepsilon(k))]^2 + W_k(t_i)[K^2(2\bar\varepsilon(k))-K^2(-2\bar\varepsilon(k))]\} + \frac{\pi^2\alpha_T}{2}F(t_f-t_i)^2$$
$$\times\Big\{\frac{\partial Y_k(t_i)}{\partial k}[K(2\bar\varepsilon(k))+K(-2\bar\varepsilon(k))] + \frac{\partial W_k(t_i)}{\partial k}[K(2\bar\varepsilon(k))-K(-2\bar\varepsilon(k))]$$
$$+ Y_k(t_i)\frac{\partial\bar\varepsilon(k)}{\partial k}[K'(2\bar\varepsilon(k))-K'(-2\bar\varepsilon(k))] + W_k(t_i)\frac{\partial\bar\varepsilon(k)}{\partial k}[K'(2\bar\varepsilon(k))+K'(-2\bar\varepsilon(k))]\Big\}$$
$$-\pi^2\alpha_T\int\frac{d\varepsilon}{2\pi}K(\varepsilon)\Big[\frac{1}{(\varepsilon+2\bar\varepsilon(k))^2}+\frac{1}{(\varepsilon-2\bar\varepsilon(k))^2}\Big]\Big\{Y_k(t_i)\Big[1 - \frac{\pi^2\alpha_T}{2}(t_f-t_i)[K(2\bar\varepsilon(k))+K(-2\bar\varepsilon(k))]\Big]$$

$$-F(t_f-t_i)\,\frac{\partial Y_k(t_i)}{\partial k}\Big\}$$

$$+\pi^2\alpha_T F(t_f-t_i)\int\frac{d\varepsilon}{2\pi}\,Y_k(t_i)\frac{\partial\bar\varepsilon(k)}{\partial k}\,\frac{\partial K(\varepsilon)}{\partial\varepsilon}$$

$$\times\left[\frac{1}{(\varepsilon-2\bar\varepsilon(k))^2}-\frac{1}{(\varepsilon+2\bar\varepsilon(k))^2}\right]. \quad \text{(A1)}$$

where $K'(x)=dK/dx$.

In the present model, with viscous friction but without a shunt resistance, the process of interaction with the thermostat changes the quasimomentum k by $k-\operatorname{sign} k$. As a result, the symmetric combination $W_k(t)$ is insensitive to the relaxation processes, and $W_k(t)=W(k-Ft)$.

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