Resonant photoabsorption and polarizability of inhomogeneous dielectric particles

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The polarizability of a small inhomogeneous particle whose dielectric constant $\varepsilon(\mathbf{r})$ crosses zero is shown to be of a resonant nature. The problem is solved exactly if $\varepsilon(r)$ is a linear function. In this case the photoabsorption and the polarizability each have at the frequency $\omega \approx 0.273\omega_0$, a resonance of width $\gamma \approx 0.076\omega_0$, where ω_0 is the maximum plasma frequency in the particle. The results are compared with those in the case of a homogeneous dielectric ball. The natural oscillations of a small dielectric particle are analyzed. The results can be used to describe the optical properties of many-electron atoms and to interpret the characteristics of finely dispersed particles and biological specimens.

Effects which accompany the scattering of light by small particles are of interest in a long list of fields, from space physics to biology and medicine.[1,2] The particular problem in the scatterinrg (or diffraction) of light by particles of finite dimensions which has been studied most comprehensively is the scattering by a homogeneous dielectric ball (the Mie problem).[1,3] The results of the solution of this problem are widely used in interpreting a variety of experiments. Landau and Lifshitz[4] attempted to go beyond the scope of the homogeneous-ball model. They analyzed the polarizability of homogeneous dielectric particles of arbitrary shape with a sharp boundary.

In the present paper, the Mie model is generalized in another direction: the electrodynamic properties of spherically symmetric but inhomogeneous[1)] particles are analyzed. The results show that the properties of these particles are quite different from those predicted by the Mie model.

I. INTRODUCTION

The scattering of a plane electromagnetic wave by an inhomogeneous particle is described by the following equation in the case in which the electric displacement and electric field in the particle can be assumed to be linear and local:

$$\mathrm{rot\,rot}\,\mathbf{E} - k^2\varepsilon(\mathbf{r},\omega)\mathbf{E} = 0,$$

$$\mathbf{E}|_{r\to\infty} = \mathbf{e}_0 e^{i\mathbf{k}\mathbf{r}} + \mathbf{f}(\mathbf{n},k)\frac{e^{ikr}}{r}, \qquad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}. \qquad (1)$$

Here $\mathbf{e}_0$ is the amplitude of the incident wave, $f(\mathbf{n},k)$ is the amplitude of the scattering in the direction $\mathbf{n} = \mathbf{r}/r$, and $\varepsilon(\mathbf{r},\omega)$ is the distribution of the dielectric constant in the particle.[2)] For a homogeneous spherical particle with a dielectric constant $\varepsilon(\omega)$, Eq. (1) reduces to a Helmholtz equation, so the problem can be solved completely.[1,3] The exact expression for the dipole polarizability in this case can be written in the form

$$\alpha(\omega) = a^3 \frac{\varepsilon(\omega)-1}{\varepsilon(\omega)+2+\Gamma(\varepsilon(\omega),q)}, \qquad q = ka. \qquad (2)$$

If the wavelength of the radiation is significantly larger than the radius of the particle, i.e., if $\lambda \gg a$ (i.e., $q \ll 1$), the function $\Gamma(\varepsilon(\omega),q)$ in (2) can be expanded in a series in q:

$$\alpha(\omega) \approx a^3[\varepsilon(\omega)-1] \Big/ \Big\{ [\varepsilon(\omega)+2] - \frac{3}{5}q^2[\varepsilon(\omega)-2] - i\frac{2}{3}q^3[\varepsilon(\omega)-1] + \ldots \Big\}. \qquad (3)$$

We thus see that the leading term of the expansion of the polarizability $\alpha(\omega)$ in powers of q is

$$\alpha_0(\omega) = a^3 \frac{\varepsilon(\omega)-1}{\varepsilon(\omega)+2}. \qquad (4)$$

Expression (4) agrees with the exact value of the static polarizability of a homogeneous ball with a dielectric constant $\varepsilon(\omega)$ (§8 in Ref. 5). That the leading term $\alpha_0(\omega)$ of the expansion of the particle polarizability $\alpha(\omega)$ in powers of a/λ can be found from the solution of the static problem is a rather general circumstance. It has been shown[6,7] that in the long-wavelength approximation a solution of scattering problem (1) can be sought as a power series in a parameter a/λ, where a is the characteristic dimension of the region in which the quantity $\varepsilon(\mathbf{r},\omega) - 1$ is nonzero. Accordingly, we write the field and scattering amplitude in the following form in a zeroth approximation:

$$\mathbf{E} = -\nabla V(\mathbf{r}), \quad \mathbf{f}(\mathbf{n},k) = k^2[\mathbf{n}[\mathbf{d}\mathbf{n}]], \qquad (5)$$

where $V(\mathbf{r})$ and $\mathbf{d}$ are determined by the electrostatics equation

$$\mathrm{div}\,[\varepsilon(\mathbf{r},\omega)\nabla V(\mathbf{r})] = 0, \qquad (6a)$$

$$V|_{r\to\infty} = -\mathbf{e}_0\mathbf{r} + \frac{\mathbf{d}\mathbf{n}}{r^2}, \qquad \mathbf{d} = \alpha_0(\omega)\mathbf{e}_0. \qquad (6b)$$

If the particle is not spherically symmetric, the polarizability $\alpha_0(\omega)$ is a tensor.

We wish to stress that Eq. (6) and expressions (5) constitute the long-wavelength approximation of the exact formulation of scattering problem (1); this approximation is valid under the condition $a \ll \lambda$.

For $\varepsilon(\omega) = -2$, the polarizability $\alpha_0(\omega)$ in (4) has a pole due to the existence of so-called natural oscillations of Eq. (6a): solutions which are localized at the particle and which decay as $r \to \infty$. Such solutions exist for certain values of $\varepsilon(\omega)$ for homogeneous particles of arbitrary shape.[4] Note, however, that these natural oscillations are an artifact of the long-wavelength approximation: the exact polarizability $\alpha(\omega)$ found from Eq. (1) is bounded by radiative-damping

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effects. These effects can be taken into account approximately by replacing $\alpha_0(\omega)$ by[8]

$$\tilde{\alpha}(\omega)=\alpha_0(\omega)\Big/\left[1-\frac{2i}{3}k^3\alpha_0(\omega)\right]. \tag{7}$$

For the particular case of a homogeneous ball, we see by substituting (4) into (7) and comparing with (3) that expression (7) does indeed give a correct description of the radiative damping [the imaginary term in the denominator in (3)], but it ignores the shift of the resonance in the polarizability. That circumstance is of no importance to us at this point.

We will be using Eq. (6) to analyze particles with an inhomogeneous distribution of the dielectric constant $\varepsilon(\mathbf{r},\omega)$. A study of the properties of such particles leads to some new effects—effects not embodied in the homogeneous Mie model.

II. GENERAL PROPERTIES OF THE POLARIZABILITY OF AN INHOMOGENEOUS PARTICLE

Before we take up a specific model, it is worthwhile to derive some general electrodynamic properties of small particles by working directly from Eq. (6). We will be omitting the argument ω from the dielectric constant $\varepsilon(\mathbf{r},\omega)$.

1. Photoabsorption cross section. Let us calculate the energy Q absorbed by a particle per unit time,

$$Q=\frac{\omega}{8\pi}\int \mathrm{Im}(\mathbf{E}^*\mathbf{D})\,d\mathbf{r}=\frac{\omega}{8\pi}\int \mathrm{Im}\,\varepsilon(\mathbf{r})\,|\nabla V|^2\,d\mathbf{r}. \tag{8}$$

Here we have used (5). Expression (8) can be integrated by parts. By virtue of Eq. (6), we are left with only an integral over a surface S of infinite radius, so the expression for Q becomes

$$Q=-\int_S \mathbf{W}\,d\mathbf{s}\,|_{r\to\infty},\qquad d\mathbf{s}=\mathbf{n}r^2\sin\theta\,d\theta\,d\varphi,$$
$$\mathbf{W}=i\frac{\omega}{16\pi}\{\varepsilon V^*\nabla V-\varepsilon^* V\nabla V^*\}. \tag{9}$$

We see thus that in our long-wavelength approximation the vector $\mathbf{W}$ serves as the energy flux density of the electromagnetic field. Using in (9) the boundary conditions of Eq. (6), and noting that $\varepsilon(\mathbf{r})|_{r\to\infty}=1$, we find

$$Q=\frac{\omega}{2}|\mathbf{e}_0|^2\,\mathrm{Im}\,\alpha_0(\omega). \tag{10}$$

Dividing this expression by the energy flux density $(c/8\pi)|\mathbf{e}_0|^2$, we find the known expression for the photoabsorption cross section (§93 in Ref. 5; Ref. 9):

$$\sigma_a(\omega)=\frac{4\pi\omega}{c}\mathrm{Im}\,\alpha_0(\omega). \tag{11}$$

Note the difference between expressions (10) and (11). Expression (10) describes the power absorbed by a particle in a uniform external field.[10] Expression (11), on the other hand, describes the cross section for the absorption of a plane electromagnetic wave by a small particle in the dipole approximation. The latter expression is valid under two conditions. The first is $a\ll\lambda$, which makes static equations (6) applicable. The second is $k^3\alpha_0(\omega)\ll 1$, which guarantees that radiative damping will play only a minor role [see the discussion of Eq. (7)].

2. Singularity of the field; plasma (Langmuir) waves. According to expression (8), the photoabsorption Q is generally determined by the imaginary part of dielectric constant $\varepsilon(\mathbf{r})$. However, Q may also be nonzero for a real $\varepsilon(\mathbf{r})$, if the field $V(\mathbf{r})$ in (8) has a singularity. In a homogeneous particle, the field will have no singularities, as has been established previously. A singularity arises in the case of an inhomogeneous particle if $\varepsilon(\mathbf{r})$ crosses zero inside the particle. We will take a more detailed look at this question for the case of a spherically symmetric $\varepsilon(\mathbf{r})$ distribution. In Eq. (6) we can separate the angular variables in this case:

$$V(\mathbf{r})=V(r)\cos\theta, \tag{12}$$

where θ is the angle between the vectors $\mathbf{r}$ and $\mathbf{e}_0$. For the radial function $V(r)$ we find

$$V''(r)+\left[\frac{2}{r}+\frac{\varepsilon'(r)}{\varepsilon(r)}\right]V'(r)-\frac{2}{r^2}V(r)=0, \tag{13a}$$

$$V(0)<\infty,\qquad V|_{r\to\infty}=-e_0 r\left(1-\frac{\alpha_0(\omega)}{r^3}\right), \tag{13b}$$

where the prime means differentiation with respect to r. Let us assume that at some point $r=r_0$ the function $\varepsilon(r)$ vanishes linearly; i.e., we assume $\varepsilon(r_0)=0$, with $\varepsilon'(r_0)\neq 0$. The point $r=r_0$ is a singular point for Eq. (13), and a general solution (13) near this point is

$$V(r)=C_1\varphi_1(\rho)+C_2[\varphi_1(\rho)\ln\rho+\varphi_2(\rho)]. \tag{14}$$

Here $\rho=r_0-r$, and the functions $\varphi_1(\rho)$ and $\varphi_2(\rho)$ are holomorphic in a certain neighborhood of the point $\rho=0$; here $\varphi_1(0)=\varphi_2(0)=1$ (Ref. 11). It can be seen from (14) that one of the two linearly independent solutions of Eq. (13) diverges logarithmically at $r=r_0$. To give expression (14) an unambiguous meaning, we must specify the rule for circumventing the branch point $r=r_0$. We use the following standard procedure for this purpose: We assume that the dielectric constant ε has a small, positive imaginary increment $\mathrm{Im}\,\varepsilon=\delta>0$. Physically, this assumption corresponds to the introduction of an infinitely slight absorption. Near the point r_0 we have

$$\varepsilon(r)\approx\varepsilon'(r_0)(r-r_0)+i\delta=\varepsilon'(r_0)(r-\tilde{r}_0),$$
$$\tilde{r}_0=r_0-i\delta/\varepsilon'(r_0).$$

The substitution $\varepsilon\to\varepsilon+i0$ is thus equivalent to assigning to r_0 an infinitely small imaginary part with a sign which is the opposite that of the derivative $(\partial\varepsilon/\partial r)_{r=r_0}$. If $\varepsilon(r)$ increases monotonically from the center of the particle to its periphery, we should set $\ln(r_0-r)=\ln|r_0-r|-i\pi\theta(r-r_0)$ in (14).

Let us find the contribution of this singularity of Eq. (13) to the photoabsorption, (8). Substituting (12) into (8), we find

$$Q=\frac{\omega}{6}\int_0^\infty \mathrm{Im}\,\varepsilon(r)\,[r^2|V'|^2+|V|^2]\,dr.$$

We again make use of the formal substitution $\varepsilon\to\varepsilon+i\delta$. The latter expression can then be rewritten as

$$Q=\frac{\omega}{6}\int_0^\infty \frac{\delta}{\varepsilon^2+\delta^2}[r^2|\varepsilon V'|^2+|\varepsilon V|^2]\,dr.$$

Using the known relation

$$\lim_{\delta\to+0}\frac{\delta}{\varepsilon^2+\delta^2}=\pi\delta(\varepsilon)$$

and a relation which follows from (14),

$$\varepsilon V|_{r\to r_0}\sim (r-r_0)\ln(r-r_0)\to 0,\quad \varepsilon V'|_{r\to r_0}=\text{const},$$

we finally find

$$Q=\frac{\pi}{6}\,\omega r_0^2\,\frac{|\varepsilon V'|^2}{|\varepsilon'|}\bigg|_{r=r_0}. \tag{15}$$

The quantity $\varepsilon(r)V'(r)$ agrees within a factor of $\cos\theta$ with the radial component of the electromagnetic displacement vector **D**; its continuity along r follows directly from the equation $\operatorname{div}\mathbf{D}=0$.

If $\varepsilon(\mathbf{r})$ does not have an imaginary part, the photoabsorption which we are discussing here corresponds physically to the excitation of plasma waves near the point with $\varepsilon=0$. This phenomenon is analogous to a linear conversion of a p-polarized transverse wave into plasma waves of a plasma which is inhomogeneous in one dimension (§88 in Ref. 5; Ref. 12). The field structures of the electromagnetic waves are completely different in these two cases, however, as we will see in Sec. III.

3. An inhomogeneous particle has no natural oscillations. Let us consider the question of whether Eq. (6a) has natural oscillations, by which we mean nontrivial solutions which decay as $r\to\infty$. Multiplying (6a) by $V^*(\mathbf{r})$ and integrating over the entire space, we can show that the natural oscillations must satisfy the condition

$$\int \varepsilon(\mathbf{r})\,|\nabla V|^2\,d\mathbf{r}=0. \tag{16}$$

It can be seen from (16) that Eq. (6a) has no natural oscillations if $\varepsilon(\mathbf{r})$ is complex with $\operatorname{Im}\varepsilon(\mathbf{r})>0$ or if $\varepsilon(\mathbf{r})$ is real and positive throughout space. Relation (16) does not, however, forbid the existence of natural oscillations for real ε if there is a region in which the relation $\varepsilon(\mathbf{r})<0$ holds. For homogeneous particles of arbitrary shape, with $\varepsilon(\mathbf{r})$ changing discontinuously at the boundary, such solutions do in fact exist at certain negative values of the dielectric constant. Their properties were studied in detail in Ref. 4. A particular case of a natural oscillation is a dipole oscillation of a homogeneous spherical particle with $\varepsilon=-2$; this oscillation gives rise to a pole in polarizability (4). We will show that in the case of an inhomogeneous particle, for which $\varepsilon(\mathbf{r})$ crosses zero continuously, Eq. (6a) has no natural oscillations.

Let us go back to the spherically symmetric case and make the assumption that we have natural oscillations of the dipole type, (12). The radial function $V(r)$ near the singular point $r=r_0$ of Eq. (13a) can then again be written in the form (14). We will show that in the case of natural oscillations the function $V(r)$ cannot have a singularity at the point $r=r_0$, i.e., that the coefficient C_2 in (14) must be zero. Specifically, as we showed in the preceding subsection, the term which contains the logarithm in (14) leads to a nonzero value of Q in (15). This value agrees within a factor of $\omega/8\pi$ with the imaginary part of the integral in (16) [cf. expressions (8) and (16)]. This result, however, contradicts condition (16), which must be satisfied by the natural oscillations. Consequently, if a natural oscillation does exist, then we have $C_2=0$ for it, and it indeed has no singularity[3)] at the point r_0.

We denote by G the region with $\varepsilon(\mathbf{r})<0$. Region G is bounded, since we have $\varepsilon(\mathbf{r})|_{r\to\infty}=1>0$. We denote its boundary by S. Multiplying (6a) by $V^*(\mathbf{r})$, and integrating over G, we find

$$\int_S \varepsilon V^*\nabla V\cdot\mathbf{n}_S\,dS-\int_G \varepsilon|\nabla V|^2\,d\mathbf{r}=0, \tag{17}$$

where $\mathbf{n}_S$ is the outward normal to S. Since we have $\varepsilon(\mathbf{r})|_S=0$, the surface integral in (17) can be nonzero only if $V(\mathbf{r})$ has a singularity on S. However, as we showed above for the case of a spherically symmetric $\varepsilon(\mathbf{r})$ distribution, natural oscillations of Eq. (6a) could not have such a singularity. We thus find that relation (17) can hold for a natural oscillation only if $V(\mathbf{r})\equiv 0$. A spherically inhomogeneous particle thus has no natural oscillation for any discontinuous $\varepsilon(\mathbf{r})$ distribution.[4)]

It can be seen from (6) that the existence of natural oscillations leads to a singularity of the polarizability $\alpha_0(\omega)$. It can be concluded from the results of this section of the paper that for an inhomogeneous particle $\alpha_0(\omega)$ will not have singularities at any real ω. We recall that a homogeneous spherical particle has natural oscillations in the case $\varepsilon=-2$. Its polarizability diverges in this case [see (4)]. A smearing of the boundary of the particle eliminates this divergence and gives rise to an absorption even if ε is real, if $\varepsilon(\mathbf{r})$ is continuous. The absorption of the incident radiation occurs in a thin surface layer.

4. Analytic properties of the polarizability of an inhomogeneous particle; sum rule. We defined $\alpha_0(\omega)$ as the coefficient in the asymptotic solution of Eq. (6) as $r\to\infty$. The analytic properties of $\alpha_0(\omega)$ as the response function must therefore be extracted from the properties of the function $\varepsilon(\mathbf{r},\omega)$. We again consider the spherically symmetric case. We can show that if $\epsilon(r,\omega)$ is sufficiently smooth along the variable r and has the ordinary analytic properties in the upper half-plane of the complex variable ω (we denote this region by I^+), i.e., if $\epsilon(r,\omega)$ has no zeros and is analytic in I^+ (§82 in Ref. 5), then $\alpha_0(\omega)$ is analytic in I^+.

In place of boundary-value problem (13), we consider a Cauchy problem, specifying the initial conditions on Eq. (13a) to be $V(r)|_{r=0}=r$ [we are assuming that $\epsilon'(r)/\varepsilon(r)$ is finite at $r=0$]. The solution of this problem (which we denote by $\tilde{V}$) will obviously be the same, within a coefficient, as the solution of problem (13). In the limit $r\to\infty$ we have

$$\tilde{V}|_{r\to\infty}=C_1(\omega)r+C_2(\omega)\frac{1}{r^2}.$$

Comparing this expression with (13b), we find $\alpha_0(\omega)=-C_2(\omega)/C_1(\omega)$. To this Cauchy problem we can apply a theorem concerning the analytic dependence of a solution of this problem on a parameter.[11] This theorem asserts that the analyticity of the coefficients of Eq. (13a) in I^+ [which is a consequence of the assumed properties of the function $\varepsilon(r,\omega)$] implies the analyticity of its solution in this region and thus the analyticity of the coefficients $C_1(\omega)$ and $C_2(\omega)$ in I^+. Furthermore, $C_1(\omega)$ cannot have zeros in I^+ [the condition $C_1(\omega)=0$ corresponds to a natural oscillation of Eq. (13a)], for otherwise fluctuations of the dipole moment at the corresponding frequency would grow exponentially, and the particle would be unstable. It follows that $\alpha_0(\omega)$ is analytic in I^+, so it satisfies the Kramers-Kronig relations.

We know that in the limit $\omega\to\infty$ the dielectric constant has the asymptotic form $\varepsilon(\mathbf{r},\omega)|_{\omega\to\infty}=1-\omega_p^2(\mathbf{r})/\omega^2$, where $\omega_p^2(\mathbf{r})=4\pi n(\mathbf{r})e^2/m$ and $n(\mathbf{r})$ is the electron density. Applying a perturbation theory in $(1-\varepsilon)$ to (6), we can easily show that this approach leads to the correct high-frequency behavior of the polarizability:

$$\alpha_0(\omega)|_{\omega\to\infty} = -\frac{Ne^2}{m\omega^2}. \quad (18)$$

Here N is the total number of electrons in the particle. Using (18), we find from the Kramers-Kronig relations that photoabsorption cross section (11) satisfies the dipole sum rule

$$\int_0^\infty \sigma_a(\omega)\,d\omega = \frac{2\pi^2 e^2}{mc} N. \quad (19)$$

III. EXACTLY SOLVABLE MODEL OF AN INHOMOGENEOUS DIELECTRIC PARTICLE

Let us assume a plasma or electron swarm[13] in which the electron density varies linearly along the radius:

$$n(r) = n_0(1 - r/a),$$

so the dielectric constant is given by

$$\varepsilon(r) = \begin{cases} 1 - \dfrac{\omega_0^2}{\omega^2}\left(1 - \dfrac{r}{a}\right), & r \leqslant a, \\ 1, & r > a, \end{cases} \quad (20)$$

where $\omega_0 = (4\pi n_0 e^2/m)^{1/2}$ is the maximum value of the plasma frequency in the particle. The function $\varepsilon(r)$ is continuous, and at frequencies $\omega < \omega_0$ it vanishes at the point

$$r_0 = a(1 - \omega^2/\omega_0^2). \quad (21)$$

Formally, we could also deal with negative values of r_0 (which would correspond to the case $\omega > \omega_0$) and also values $r_0 > a$ (a dielectric particle with $\varepsilon > 1$). Figure 1 shows the function $\varepsilon(r)$ in these three cases.

The polarizability of a plasma particle of this sort is given by Eq. (13), which takes the following form when we use (20):

$$V''(r) + \left(\frac{2}{r} + \frac{1}{r - r_0}\right)V'(r) - \frac{2}{r^2}V(r) = 0,$$
$$V(0) < \infty, \quad V|_{r\geqslant a} = -e_0 r\left(1 - \frac{\alpha_0(\omega)}{r^3}\right). \quad (22)$$

Introducing the dimensionless coordinate $x = r/r_0$, and using the substitution $V = x\varphi(x)$, we can put Eq. (22) in the form of the hypergeometric equation, so that a solution, bounded at $r = 0$, of Eq. (22) in the region $r \leqslant a$ can be expressed in terms of the hypergeometric function:

$$V(r) = A\frac{r}{r_0}F\left(\tilde{a}, b, c; \frac{r}{r_0}\right), \quad r \leqslant a,$$
$$\tilde{a} = 2 + 3^{1/2}, \quad b = 2 - 3^{1/2}, \quad c = 4. \quad (23)$$

The constant A in this expression and also the quantity $\alpha_0(\omega)$ in (22) are found from the conditions for the joining of function (23) with the expression for $V(r)$ at $r > a$ [see (22)]. Using the known formula for the differentiation of the hypergeometric function, we find the following expressions for A and $\alpha_0(\omega)$:

$$A = \frac{-12 e_0 r_0 x_0}{F_2 + 12 x_0 F_1}, \quad \alpha_0(\omega) = \frac{a^3}{1 + 12 x_0 F_1/F_2},$$
$$F_1 = F\left(\tilde{a}, b, c; \frac{1}{x_0}\right), \quad F_2 = F\left(\tilde{a}+1, b+1, c+1; \frac{1}{x_0}\right), \quad x_0 = \frac{r_0}{a}. \quad (24)$$

It can be seen from these expressions that the dimensionless quantity $\alpha_0(\omega)/a^3$ is determined by the value of the one parameter $x_0 = r_0/a$, which is related in a single-valued way with the dimensionless frequency $\Omega = \omega/\omega_0$ by

$$x_0 = 1 - \Omega^2. \quad (25)$$

Let us consider various ranges of this parameter.

At $x_0 < 0$ (i.e., at frequencies ω above the maximum plasma frequency ω_0) and at $x_0 > 1$ (i.e., at $r_0 > a$, for a dielectric particle with $\varepsilon > 1$), the dielectric constant is positive everywhere inside the particle, and the polarizability $\alpha_0(\omega)$ is real, as can be seen from (24). For our purposes, the interesting case is $0 \leqslant x_0 \leqslant 1$ (i.e., $\omega \leqslant \omega_0$), in which $\varepsilon(r)$ crosses zero inside the particle. In this case the function (23) diverges logarithmically at the point $r = r_0$. Our values of the hypergeometric parameters $\tilde{a}$, b, and c thus correspond to the special case $\tilde{a} + b = c$, in which the representation of the hypergeometric function in the region $|1 - x| < 1$ is[14]

$$F(a, b, a+b; x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(k!)^2}$$
$$\times[2\psi(k+1) - \psi(a+k) - \psi(b+k) - \ln(1-x)](1-x)^k,$$
$$|1 - x| < 1, \quad (26)$$

where $\psi(x)$ is the logarithmic derivative of the gamma function. At $x > 1$, the function (26) is complex, so the values of

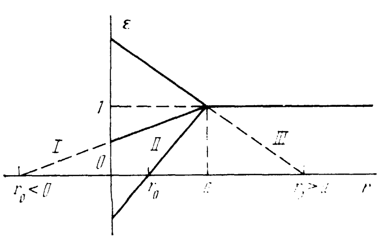

FIG. 1. Dielectric constant of an inhomogeneous plasma, (20). I—$r_0 < 0$ (i.e., $\omega > \omega_0$); II—$0 \leqslant r_0 \leqslant a (\omega < \omega_0)$; III—$(r_0 > a)$ inhomogeneous dielectric particle with $\varepsilon > 1$.

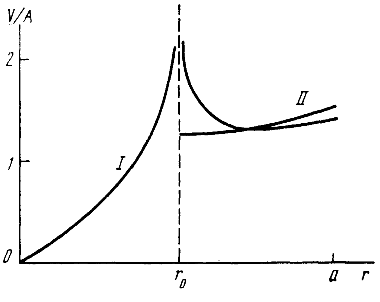

FIG. 2. The potential distribution $V(r)$ [normalized to the constant A; see (23) and (24)] inside a plasma in a uniform oscillatory external field. I, II—Real and imaginary parts of the function $V(r)/A$ in the case $r_0 = a/2$.

A and $\alpha_0(\omega)$ in (24) turn out to be complex in the case $\omega < \omega_0$.

It can be seen from expressions (23) and (26) that the function $V(r)$ takes indeed the form (14) near the singular point $r = r_0$. Figure 2 shows a plot of $V(r)/A$ for the case $r_0 = a/2$ (here we have $\omega = \omega_0/2^{1/2}$). At $r < r_0$, this quantity is real, and it diverges logarithmically at the point $r = r_0$ [the potential $V(r)$ itself in (23) is complex at $0 < r_0 < a$ for all values of r, since the coefficient A is complex], while at $r \geqslant r_0$ its imaginary part acquires a discontinuity.

As we mentioned in the preceding section of the paper, the effects which we are discussing here are analogous to a linear conversion of transverse waves into plasma waves in a plasma which is inhomogeneous in one dimension. It is accordingly interesting to compare the field structures in the two cases. Figure 3, a and b, makes this comparison on a qualitative level. Far from the critical point $\varepsilon = 0$ the field structures are different, as can be seen from Fig. 3, but near the critical point the nature of the singularity is precisely the same in the two cases. Note that, in contrast with the case of an inhomogeneous spherical particle which we have been discussing, there is no exactly solvable model of linear wave conversion in a plasma which is inhomogeneous in one dimension.

Figure 4 shows the polarizability $\tilde{\alpha} = \alpha_0(\omega)/a^3$ as a function of the parameter $\Omega = \omega/\omega_0$. Interestingly, the frequency dependence Im $\tilde{\alpha}(\Omega)$ and Re $\tilde{\alpha}(\Omega)$ looks the same as it would if we were dealing with an absorption line having a frequency $\omega/\omega_0 = 0.273$ and a width $\gamma/\omega_0 = 0.076$. Near the resonant frequency there is a region of negative dispersion $[d\,\mathrm{Re}\,\alpha_0(\omega)/d\omega < 0]$, in accordance with the Kramers-Kronig relations and as in the case of ordinary absorption lines. These properties also characterize our model of an inhomogeneous dielectric particle, in contrast with the case of ordinary absorption lines. These properties also characterize our model of an inhomogeneous dielectric particle, in contrast with the case of a homogeneous particle, which has no absorption at all if ε is real.

At $\omega \geqslant \omega_0$ the imaginary part of the polarizability vanishes, while the real part takes on its asymptotic form (18) quite rapidly [at $\omega = \omega_0$, the exact value is $\tilde{\alpha} = -1/(5 + 3^{3/2}) \approx -0.098$, while (18) predicts $-1/12 \approx -0.083$]. The case $\omega = 0$ corresponds to a homogeneous dielectric ball with $\varepsilon = -\infty$: $\tilde{\alpha}(0) = 1$.

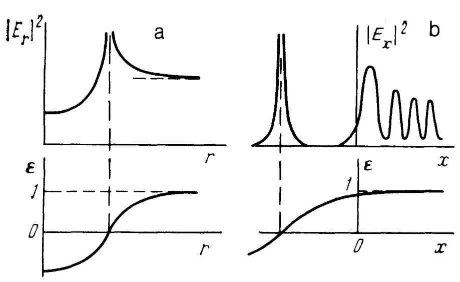

FIG. 3. Qualitative field structure of the electromagnetic wave (a) in the case of scattering by a small inhomogeneous spherical particle and (b) in the case of reflection of a p-polarized wave from a plasma boundary which is inhomogeneous in one dimension (the wave is incident from the right in this figure). In both cases the dielectric constant crosses zero, giving rise to a singularity in the field, which is associated with the excitation of plasma waves. The field components along the gradient of ε are shown here. The point $x = 0$ in Fig. 3b corresponds to a "cutoff" of the field.[12]

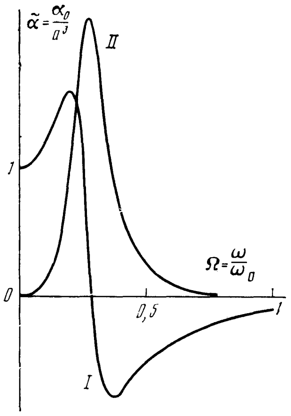

FIG. 4. Frequency dependence of the polarizability of a plasma, (20), normalized to a^3. I, II—Real and imaginary parts of the quantity $\tilde{\alpha}(\Omega) = \alpha_0(\omega)/a^3$; at $\omega > \omega_0$, we have Im $\alpha_0(\omega) = 0$.

IV. CONCLUSION

An exactly solvable model of an inhomogeneous dielectric particle has been proposed. It has been shown that this model has absorption even if the dielectric constant $\varepsilon(\mathbf{r})$ is real. For a plasma, with $\varepsilon = 1 - \omega_p^2(\mathbf{r})/\omega^2$, the frequency dependence of the photoabsorption is a resonant curve. The corresponding polarizability and cross section of the photoabsorption satisfy the Kramers-Kronig relations and a sum rule.

It has been shown that an inhomogeneous dielectric particle has no natural oscillations for any continuous distribution $\varepsilon(\mathbf{r})$, in contrast with the case of a homogeneous spherical particle, which does have natural dipole oscillations in the case $\varepsilon = -2$.

The methods and results of this study can be used to calculate the optical characteristics of complex atoms in the vacuum-UV and soft x-ray regions and in research on clusters, finely divided particles, biological specimens, and plasmas in the atmosphere and in space.

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[1)] We have in mind particles whose dielectric-constant profile varies continuously.

[2)] Everywhere except in Subsection II.3, the frequency ω is assumed to be real.

[3)] This assertion is valid for a natural oscillation of any multipolarity since the type of singularity of the equation for the radial function $V(r)$ and also expression (14) remain the same as in the dipole case, which we are discussing.

[4)] It can be shown that this assertion remains valid in the absence of spherical symmetry.

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