Mesoscopic behavior of Josephson junctions with randomly disposed Abrikosov vortices

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The dependence, on the external magnetic field, of the critical current of a Josephson junction with irregularly disposed Abrikosov vortices (AV) whose axes are parallel to the junction plane are calculated. The $I_c(H)$ dependence should show both periodic and random ("mesoscopic") oscillations with amplitudes of order of the current itself. This mesoscopic behavior is attributed to the randomness of the Josephson phase in the junction. The characteristic values of the amplitude and of the period of the oscillations are obtained, and the region in which the effect exists is also determined.

1. INTRODUCTION

Josephson junctions containing Abrikosov vortices (AV) have been attracting great interest of late.[1–5] In such junctions the dependence of the critical current on the magnetic field changes when the vortex density is increased, and can differ greatly from the usual "Fraunhofer" dependence.

In Refs. 1–3 were considered Josephson junctions with AV having axes almost perpendicular to the junction plane, so that these AV could be regarded as "microresistors" (regions with greatly decreased current). Vortices are usually pinned to defects contained in superconductors (grain boundaries, normal-phase inclusions), and are therefore randomly disposed. The dependence of the critical current on the magnetic field of disordered junctions with microshorts and microresistors was considered theoretically in Ref. 6, where it was shown that random oscillations of the critical current, i.e., a mesoscopic behavior of the Josephson junctions, should be observed in strong-field regions.

The present paper deals with a Josephson junction with randomly disposed AV whose axes are parallel to the junction plane. The mean current (averaged over different samples) was found for such structures in Ref. 7. Starting with a sufficiently low vortex density $n_0 \sim 1/\lambda L$ (L is the junction length and λ is the depth of penetration of the magnetic field in the superconductors), the dependence of the mean current on the magnetic field ceases to be periodic:

$$\overline{I_c^2}=2j_0^2Ln\lambda\alpha_1\{(n\lambda\alpha_1)^2+[2\pi(\Phi/\Phi_0L-\alpha_2n\lambda)]^2\}^{-1},\quad 1/\lambda L\ll n\ll 1/\lambda^2, \tag{1}$$

where $\Phi = 2HL\lambda$ is the magnetic flux, Φ_0 is the magnetic-flux quantum, j_0 is the critical-current density, and α_1 and α_2 are quantities on the order of unity.

At a vortex density $n > (\lambda L)^{-1}$, however, as will be shown in Secs. 2 and 3, the critical current can vary greatly from sample to sample and (in analogy with the results of Ref. 6) differ from the mean current obtained in Ref. 7. We shall therefore obtain in Sec. 2 the probability of fluctuations of the critical current of a Josephson junction with AV, and determine in Sec. 3 the correlation function of the current as the external magnetic field is varied.

2. PROBABILITY $P(I_c^2)$ OF CRITICAL-CURRENT FLUCTUATIONS IN A JOSEPHSON JUNCTION WITH AV

Consider a small Josephson junction $L<\lambda_j$ (λ_j is the Josephson penetration depth) in a magnetic field H parallel to the junction plane. The junction contains randomly placed AV whose axes are also parallel to the junction plane. The critical current is then determined by the usual equation[8]

$$I_c^2=j_0^2\left|\int_0^L dx\exp[i\varphi(x)]\right|^2. \tag{2}$$

The phase difference φ depends on the external field and on the random coordinates (x_i, y_i) of the AV in the banks of the junction[5,7]

$$\varphi=\sum_{i=1}^N\left(-\frac{2}{\lambda}\right)\int_0^x Z(x-x_i,y_i)\,dx+\frac{2\pi\Phi x}{\Phi_0 L},$$

$$Z(x-x_i,y_i)=|y_i|[(x-x_i)^2+y_i^2]^{-1/2}K_1\left(\frac{[(x-x_i)^2+y_i^2]^{1/2}}{\lambda}\right), \tag{3}$$

where $K_i(x)$ is a modified Bessel function.

Since the phase $\varphi(x)$ is random, the critical current varies from sample to sample. To determine the experimentally observable critical current it is necessary therefore to calculate the probability $P(I_c)$ that the junction with the AV is characterized by a critical current I_c.

The probability density $P(I_c^2)$ is determined by the functional integral

$$P(I_c^2)=\int D\varphi\rho(\varphi)\delta\left(I_c^2-j_0^2\left|\int_0^L dx\exp(i\varphi(x))\right|^2\right), \tag{4}$$

where $\rho(\varphi)$ is the probability that a phase distribution $\varphi(x)$ will be produced in the junction. Assuming the AV to be randomly distributed, we reduce the functional integral to the usual multiple one (S is the junction area)

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$$P(I_c^2)=\prod_{i=1}^{N}\int\frac{dx_i\,dy_i}{S}\prod_{j=1}^{L/\Delta x}\int d\varphi_j\delta\Big(\varphi_j-\varphi_{j-1}+\frac{2}{\lambda}\sum_{i=1}^{N}Z(x_j-x_i,y_i)\Delta x-\frac{2\pi\Phi\Delta x}{\Phi_0 L}\Big)\delta\Big(I_c^2-(\Delta x)^2j_0^2\Big[\sum_j\exp(i\varphi_j)\Big]\times\Big[\sum_j\exp(-i\varphi_j)\Big]\Big). \qquad (5)$$

In the final solution Δx must be made to tend to zero.

Following Ref. 6, we transform (5), with the aid of Fourier transformation of the δ functions, into

$$P(I_c^2)=\prod_{i=1}^{N}\int\frac{dx_i\,dy_i}{S}\prod_{j=1}^{L/\Delta x}\int_{-\infty}^{\infty}d\varphi_j\int_{-\infty}^{\infty}\frac{dt_j}{2\pi}\times\exp\Big\{i\sum_j t_j\Big[\varphi_j-\varphi_{j-1}-\frac{2\pi\Phi\Delta x}{\Phi_0L}+\frac{2}{\lambda}\sum_{i=1}^{N}Z(x_j-x_i,y_i)\Delta x\Big]\Big\}\int\frac{ds\,dt\,du}{(2\pi)^2}\times\exp\Big\{isI_c^2+itu-i\Delta xj_0\sum_j[st\exp(i\varphi_j)+u\exp(-i\varphi_j)]\Big\}. \qquad (6)$$

The integrals with respect to x_i and y_i can be evaluated by the procedure of Ref. 7. We obtain then

$$P(I_c^2)=(2\pi)^{-2}\int ds\,dt\,du\exp(isI_c^2+itu)\times\prod_{j=1}^{L/\Delta x}\int_{-\infty}^{\infty}d\varphi_j\exp\Big\{-i\Delta xj_0\Big[\sum_j(st\exp(i\varphi_j)+u\exp(-i\varphi_j))\Big]\Big\}\int_{-\infty}^{\infty}\frac{dt_j}{2\pi}G(t_j)\times\exp\Big\{i\sum_j t_j\Big(\varphi_j-\varphi_{j-1}-\frac{2\pi\Phi\Delta x}{\Phi_0L}\Big)\Big\},$$

$$G(t_j)=\exp n\int dx\,dy\Big[\exp\Big(\frac{2}{\lambda}i\sum_j t_jZ(x_j-x,y)\Big)-1\Big]. \qquad (7)$$

Changing the places of the integrals with respect to φ_j and t_j and calculating the integrals with respect to φ_j we obtain

$$P(I_c^2)=(2\pi)^{-2}\int ds\,dt\,du\,P(s,t,u)\exp(isI_c^2+itu),$$

$$P(s,t,u)=\prod_{j=1}^{L/\Delta x}\int_0^{2\pi}d\varphi_j\exp\Big\{-i\Delta xj_0\Big[\sum_j(st\exp(i\varphi_j)+u\exp(-i\varphi_j))\Big]+i\Big[\sum_j\varphi_j(t_j-t_{j-1})-t_j\frac{2\pi\Phi\Delta x}{\Phi_0L}\Big]\Big\}\times\int\frac{dt_j}{2\pi}G(t_j)\Big\{\sum_{k=-\infty}^{\infty}\exp[2\pi ik(t_j-t_{j-1})]\Big\}$$

$$=\prod_{j=1}^{L/\Delta x}\oint\frac{dz_j}{iz_j}\int\frac{dt_j}{2\pi}\exp\Big\{-i\Delta xj_0\sum_j stz_j+uz_j^{-1}\Big\}\times z_j^{t_j-t_{j-1}}\Big(\sum_{n=-\infty}^{\infty}\delta(t_j-t_{j-1}-n)\Big)\exp\Big(-i\sum_j t_j\frac{2\pi\Phi\Delta x}{\Phi_0L}\Big). \qquad (8)$$

Evaluating the integrals with respect to z_j we obtain finally

$$P(s,t,u)=\prod_{j=1}^{L/\Delta x}\sum_{n_j=-\infty}^{\infty}\int dt_j\delta(t_j-t_{j-1}+n_j)\times J_{t_j-t_{j-1}}(2\Delta xj_0(stu)^{1/2})\Big(\frac{u}{st}\Big)^{(t_j-t_{j-1})/2}\times G(t_j)\exp\Big(-i\sum_j t_j\frac{2\pi\Phi\Delta x}{\Phi_0L}\Big), \qquad (9)$$

where $J_n(x)$ is a Bessel function.

In the limit as $\Delta x\to0$ in Eq. (9), n_j can take on only the values 0 and ±1, and the $t(x)$ dependence takes then the form shown in Fig. 1. We consider hereafter a Josephson junction with a sufficiently large AV density $n\gg(\lambda L)^{-1}$. At such densities the function $G(t)$ is strongly attenuated with increase of t, and the contribution made to the probability by a function $t(x)$ of the type shown dashed in Fig. 1 is small. On the other hand, the contribution to P from functions $t(x)$ such that t_j take on values 0 and ±1 [such $t(x)$ are shown in Fig. 1 by thick lines] can be easily reduced to the form

$$P(s,t,u)=\sum_{N=0}^{\infty}(-2j_0^2stu)^N\int_0^L dx_1\int_{x_1}^L dx_2\ldots\int_{x_{2N-1}}^L dx_{2N}\times\mathrm{Re}\Big\{G(x_2-x_1)\times\exp\Big[i\frac{2\pi\Phi}{\Phi_0L}(x_2-x_1)\Big]\Big\}\ldots\mathrm{Re}\Big\{G(x_{2k}-x_{2k-1})\times\exp\Big[i\frac{2\pi\Phi}{\Phi_0L}(x_{2k}-x_{2k-1})\Big]\Big\}, \qquad (10)$$

$$G(u)=\exp\Big[n\int dx\,dy\Big[\exp\Big(-i\frac{2}{\lambda}\int_0^u Z(t-x,y)\,dt\Big)-1\Big].$$

It follows from (8) and (10) that the coefficient of stu in the expression for the probability determines the average critical current of the junction

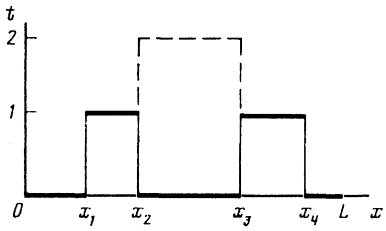

FIG. 1. Possible forms of the $t(x)$ dependence in Eq. (9) for $N=2$.

$$\overline{I_c^2}=j_0^2\int_0^L dx_1\int_{x_1}^L dx_2 Q(x_2-x_1),$$
$$Q(\tau)=2\,\mathrm{Re}\left\{G(\tau)\exp\left(\frac{2\pi i\Phi}{\Phi_0 L}\tau\right)\right\}. \quad (11)$$

This equation coincides with the expression obtained for the critical current in Ref. 7. It is seen from Eqs. (1) and (11) that the average critical current decreases monotonically when the magnetic field is increased.

To find the fluctuations of the critical current and hence the probability P, we take the Laplace transform of Eq. (10):

$$\tilde P(q)=\int_0^\infty dL\exp(-qL)P(s,t,u)$$
$$=\sum_{N=0}^\infty\int_0^\infty dx_1\int_{x_1}^\infty dx_2\ldots\int_{x_{2N-1}}^\infty dL\exp(-qL)$$
$$\times\prod_{k=1}^N Q_k(x_{2k}-x_{2k-1})(-j_0^2stu)^N. \quad (12)$$

It follows hence that

$$\tilde P(q)=\sum_{N=0}^\infty\left(\frac{1}{q}\right)^{N+1}\left(\int_0^\infty d\tau\exp(-q\tau)Q(\tau)\right)^N(-j_0^2stu)^N. \quad (13)$$

As a result we get from (8), (11), and (13)

$$P(I_c^2)=\sum_{N=0}^\infty\int\frac{ds\,dt\,du}{(2\pi)^2}\exp(isI_c^2+itu)(-j_0^2stu)^N$$
$$\times\int\frac{dq\exp(qL)}{2\pi iq^{N+1}}\left[\int_0^\infty d\tau Q(\tau)\exp(-q\tau)\right]^N$$
$$=\int\frac{dq\exp(qL)}{2\pi iq}\int_{-\infty}^\infty\frac{ds}{2\pi}\int_0^\infty dx\frac{\exp(-x+isI_c^2)}{1+\gamma isxj_0^2},$$
$$\gamma=\int_0^\infty\frac{d\tau}{q}[\exp(-q\tau)]Q(\tau). \quad (14)$$

The main contribution is made to (14) by small values of q, therefore $\gamma\sim 1/q$. Calculating the integrals with respect to q, s, and x we get

$$P(I_c^2)=\frac{1}{I_1^2}\exp\left(-\frac{I_c^2}{I_1^2}\right),$$
$$I_1^2=j_0^2L\int d\tau\,Q(\tau). \quad (15)$$

From expression (15) for the probability $P(I_c)$ it follows that the fluctuations of the critical current among the samples have the same value as the average critical current, and therefore the dependence of the critical current on the magnetic field is not determined by the average values alone. To find the real experimental dependence of the critical current on the magnetic field we must calculate the correlator

$$K(\Delta\Phi)=\langle I_c^2(\Phi)I_c^2(\Phi+\Delta\Phi)\rangle$$
$$-\langle I_c^2(\Phi)\rangle\langle I_c^2(\Phi+\Delta\Phi)\rangle.$$

3. CORRELATOR OF THE CRITICAL CURRENTS UPON VARIATION OF THE EXTERNAL MAGNETIC FIELD

We write down first the expression for the two-current correlation function

$$P(I_1,I_2)=\prod_{i=1}^N\int\frac{dx_i\,dy_i}{S}\prod_{j=1}^{L/\Delta x}\int d\varphi_j\delta\Big(\varphi_j-\varphi_{j-1}$$
$$+\sum_{i=1}^N\frac{2}{\lambda}Z(x_j-x_i,y_i)\Delta x-\frac{2\pi\Phi\Delta x}{\Phi_0 L}\Big)$$
$$\times\delta\Big(I_1^2-(\Delta x)^2j_0^2\Big[\sum_j\exp(i\varphi_j)\Big]$$
$$\times\Big[\sum_j\exp(-i\varphi_j)\Big]\Big)\delta\Big(I_2^2-(\Delta x)^2j_0^2\Big[\sum_j\exp(i\varphi_j+i\alpha\Delta xj)\Big]$$
$$\times\Big[\sum_j\exp(-i\varphi_j+i\alpha\Delta xj)\Big]\Big), \quad (16)$$

where $\alpha=\Delta\Phi/\Phi_0L$.

In analogy with Sec. 2, we take in Eq. (16) the Fourier transforms of the δ functions and calculate the integrals with respect to x_i, y_i, and φ_j:

$$P(I_1,I_2)=\int ds_1\,dt_1\,du_1\,ds_2\,dt_2\,du_2\,P(s_1,t_1,u_1;s_2,t_2,u_2)\cdot$$
$$\times\exp(is_1I_1^2+is_2I_2^2+it_1u_1+it_2u_2),$$

$$P(s_1,t_1,u_1;s_2,t_2,u_2)$$
$$=\prod_{j=1}^{L/\Delta x}\sum_{n_j=-\infty}^{\infty}\int dt_j\,\delta(t_j-t_{j-1}+n_j)J_{t_j-t_{j-1}}(2\Delta xj_0[s_1t_1$$
$$+s_2t_2\exp(i\alpha x_j)]^{1/2}[u_1+u_2\exp(-i\alpha x_j)]^{1/2})$$
$$\times G(t_j)\Big[\exp\Big(-i\sum_j t_j\frac{2\pi\Phi\Delta x}{\Phi_0L}\Big)\Big]$$
$$\times\left[\frac{s_1t_1+s_2t_2\exp(i\alpha x_j)}{u_1+u_2\exp(-i\alpha x_j)}\right]^{(t_j-t_{j-1})/2}. \quad (17)$$

To find the correlator $\langle I_1^2(\Phi)I_2^2(\Phi+\Delta\Phi)\rangle$ we must calculate the coefficient of the term $s_1t_1u_1s_2t_2u_2$. Using the fact that Δx in (17) tends to zero, we obtain again $n_j = 0, \pm 1$. The coefficient of $s_1 t_1 u_1 s_2 t_2 u_2$ is determined by the contribution from the function $t(x)$ shown by a thick line in Fig. 1. The sought correlator can be written in the form

$$K(\Delta\Phi) = \langle I_1^2(\Phi) I_2^2(\Phi+\Delta\Phi)\rangle - \langle I_1^2(\Phi)\rangle\langle I_2^2(\Phi+\Delta\Phi)\rangle$$
$$= \int_0^L dx_1 \int_{x_1}^L dx_2 \int_{x_2}^L dx_3 \int_{x_3}^L dx_4\, G(x_2-x_1) G(x_4-x_3) \cdot$$
$$\times 2\,\mathrm{Re}\left\{\left[\exp\left(ix_1\alpha - i(x_2-x_1)\frac{2\pi\Phi}{\Phi_0 L}\right) + \exp\left(ix_2\alpha + i(x_2-x_1)\frac{2\pi\Phi}{\Phi_0 L}\right)\right]\right.$$
$$\left.\times\left[\exp\left(-ix_4\alpha - i(x_4-x_3)\frac{\Phi 2\pi}{\Phi_0 L}\right) + \exp\left(-ix_3\alpha + i(x_4-x_3)\frac{\Phi 2\pi}{\Phi_0 L}\right)\right]\right\}. \quad (18)$$

Taking the Laplace transform of (18) with respect to L, we obtain for the correlator $K(\Delta\Phi)$ the expression

$$K(\Delta\Phi) = \int \frac{dq}{2\pi i q}\frac{e^{qL}}{(q^2+\alpha^2)}\left[2 j_0^2\,\mathrm{Re}\int d\tau\, Q(\tau) e^{-q\tau}\right]^2. \quad (19)$$

In the derivation of (19) we used the fact that $\Delta\Phi \ll \Phi$. Easily calculating the integral with respect to q, we obtain ultimately

$$K(\Delta\Phi) = I_1^4(\Phi)\frac{\sin^2(\Delta\Phi/\Phi_0)}{(\Delta\Phi/\Phi_0)^2}. \quad (20)$$

It follows from (20) that the correlator oscillates as a function of the magnetic flux with a period Φ_0, and decreases when $\Delta\Phi$ is increased. This leads to periodic oscillations (with period Φ_0) of the critical current of the Josephson junction as a function of the external magnetic flux, as well as to random ("mesoscopic") oscillations of $I_c(H)$. These oscillations have a random phase and vanish when averaged over different samples. This behavior of a Josephson junction with AV recalls the mesoscopic dependence of the conductivity of small metallic samples on the magnetic field.[10]

4. DISCUSSION OF RESULTS

The results show that the characteristic mesoscopic phenomena previously predicted for disordered metals[10] should be manifested in Josephson junctions containing AV whose axes are parallel to the junction plane. First of all, close to AV densities $n > (\lambda L)^{-1}$ the junction critical current fluctuates from sample to sample by an amount equal to the current itself. The critical current of a typical sample is determined at low AV densities $n\lambda^2 \ll 1$ by Eq. (1) and at high densities it takes in accordance with Eq. (6) and Ref. 7 the form

$$\overline{I_c^2} = \begin{cases} \dfrac{j_0^2 L\lambda\pi^{1/2}}{(n\pi\lambda^2\ln(n\pi\delta^2))^{1/2}}\left\{1 - \dfrac{(\pi\Phi/\Phi_0 L - 2n\pi\lambda)^2}{2\pi n\ln(n\pi\lambda^2)}\right\}, \\ \left|\dfrac{\Phi}{\Phi_0} - 2n\lambda L\right| \ll L[n\ln(\pi n\lambda^2)]^{1/2}, \\ \dfrac{j_0^2 L\pi^2 n}{(\Phi/\Phi_0 L - 2n\lambda)^3}, \quad \left|\dfrac{\Phi}{\Phi_0} - 2n\lambda L\right| \gg L[n\ln(\pi n\lambda^2)]^{1/2}. \end{cases} \quad (21)$$

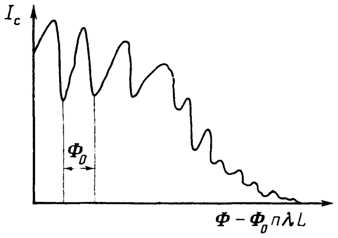

FIG. 2. Dependence of the critical current on the magnetic field of a Josephson junction with AV.

The reason for the large difference between (21) and the usual "Fraunhofer" dependence is that the current densities correlate not on the entire junction plane, but only at a small distance $r_0 \ll L$ ($r_0 \sim 1/n\lambda$) if $n\lambda^2 \ll 1$ and $r_0 \sim 1/n^{1/2}$ if $n\lambda^2 \gg 1$, and this decreases the typical current and changes its magnetic dependences. It can also be seen from (1) and (21) that the maximum of the critical current shifts towards magnetic fields $\Phi_1 \sim \Phi_0 n\lambda L$, whereas in an ordered "small" junction the critical-current maximum is at $\Phi = 0$ (Fig. 2).

It follows from our results [Eq. (20)] that the dependence of the critical current on the magnetic field should be subject to periodic oscillations (with a characteristic amplitude $\sim I_1$ that attenuates with increase of the magnetic field, and with a period Φ_0), as well as random oscillations having the same period. To observe these oscillations it is necessary that the density of the AV with axes parallel to the junction plane not be too small: $n \gg n_1 = (\lambda L)^{-1}$. Thus at AV densities $n \sim n_1$ the sample becomes mesoscopic and the number of vortices and the change of their positions can be determined from the dependence of the critical current on the magnetic field.

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