Stochastic instability of stabilized explosion in relativistic-beam–plasma system with chaotic inhomogeneities

V. V. Tamoĭkin, S. M. Faĭnshteĭn, and A. G. Fakeev

Scientific-Research Radiophysics Institute, Gorki
(Submitted 24 August 1989)
Zh. Eksp. Teor. Fiz. **97**, 1159–1165 (April 1990)

A four-wave mixing process is analyzed in a system consisting of a relativistic beam and a plasma whose electron density has chaotic inhomogeneities. Two of the waves are electromagnetic, while the other two are longitudinal; one of the longitudinal waves has a positive energy, one a negative energy. The velocity dependence of the mass of the beam electrons is known to lead to terms corresponding to a nonlinear frequency shift in the equations for the slowly varying complex amplitudes in such a system. The appearance of these terms leads in turn to saturation of the explosive instability. A system of stochastic differential equations in Itô form is constructed for a medium with inhomogeneities. A solution is derived for intermediate wave intensities. This solution indicates an exponential growth with increasing length of the region of the nonlinear interaction. The instability observed is termed "stochastic." It results from the diffusion of phase trajectories, itself a consequence of the random phase mismatch of the waves.

An explosive instability can occur in a beam-plasma system if high-frequency waves have a negative energy, and low-frequency waves a positive energy (Ref. 1, for example). This instability reaches saturation if a cubic nonlinearity is incorporated in the initial equations, as the result of either a nonlinear frequency shift or a nonlinear absorption.[1] The explosive instability in a system consisting of a relativistic electron beam and a "cold" plasma was studied in Ref. 2. It was shown that the nonlinear dependence of the mass of the beam electrons on their velocity leads to dynamic saturation of this instability, as the result of nonlinear four-wave mixing. Two of the waves here are plasma waves, one with a negative energy and one with a positive energy, while the two other waves are electromagnetic. It was also pointed out in Ref. 2 that direct conversion of beam energy into electromagnetic radiation could occur (it was assumed that the waves were propagating along the direction of the beam velocity $\mathbf{V}_0$). This conversion would be extremely important for practical applications in plasma electronics and astrophysics.

Also clearly of interest is an analysis of the stability of an explosion which has already been stabilized by a nonlinear frequency shift in a medium with steady-state random inhomogeneities which cause a linear random deviation from matching in terms of wave numbers (or phases). A new instability in this nonequilibrium system is the subject of the present paper. We call it a "stochastic" instability. We show that the diffusive spreading of phase trajectories causes a destabilization of the explosive instability. In other words, the "explosion" is not shut off by a nonlinear frequency shift: The average wave intensities grow exponentially with distance. This result is important from the standpoint of both the general theory of nonlinear waves and practical applications, since the level of electromagnetic waves which are excited may, by virtue of the randomization of phases, be significantly higher than that in a corresponding system without fluctuations.

1. EQUATIONS DESCRIBING THE WAVE INTERACTION

As the initial system of equations we use Maxwell's equations and quasihydrodynamic equations for the plasma electrons and for the relativistic monoenergetic beam:

$$\operatorname{rot}\mathbf{H} = -\frac{4\pi e}{c}n\mathbf{v} - \frac{4\pi e}{c}n_s\mathbf{v}_s + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \qquad \operatorname{div}\mathbf{H}=0,$$

$$\operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \qquad \operatorname{div}\mathbf{E} = -4\pi e(n+n_s-n_0),$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{e}{m}\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right), \tag{1}$$

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s\nabla)\mathbf{v}_s = -\frac{e}{m}\gamma_0^{-1}\left\{\mathbf{E} + \frac{1}{c}[\mathbf{v}_s\mathbf{H}] - \frac{\mathbf{v}_s}{c^2}(\mathbf{v}_s\mathbf{E})\right\},$$

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\mathbf{v}) = 0,$$

$$\frac{\partial n_s}{\partial t} + \operatorname{div}(n_s\mathbf{v}_s) = 0.$$

Here n_0 is the density of the ions whose charge neutralizes that of the electrons. We can ignore the motion of the ions in the high-frequency case discussed below. Also, n, n_s, $\mathbf{v}$, and $\mathbf{v}_s$ are the densities and velocities of the plasma and beam electrons, respectively, $\gamma_0 = (1 - \mathbf{v}_s^2/c^2)^{-1/2}$, and c is the velocity of light.

In the linear approximation, we can easily derive from (1) dispersion relations for plasma waves and electromagnetic waves which are propagating along the direction of the unperturbed beam velocity $\mathbf{V}_0 \| x$:

$$1 - \frac{\omega_0^2}{\Omega^2} - \frac{\omega_{0s}^2}{(\Omega - qV_0)^2} = 0,$$

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_0^2}{\omega^2} - \frac{\bar{\omega}_{0s}^2}{\omega^2}\right). \tag{2}$$

Here $\omega_{0s}^2 = 4\pi e^2 N_s/m\gamma^3$, $\omega_0^2 = 4\pi e^2 N/m, \gamma = (1 - V_0^2/c^2)^{-1/2}$, N and N_s are the equilibrium densities of the plasma and beam electrons, $\bar{\omega}_{0s}^2 = 4\pi e^2 N_s/m\gamma$ ($\gamma \gg 1$, $V_0 \lesssim c$), and m is the rest mass of an electron. It was shown in Ref. 2 that when cubic nonlinear terms are taken into account in Eqs. (1) the following decay process can occur:

$$\Omega_2 = \Omega_1 + \omega + \omega, \quad q_2 = q_1 + k + k. \tag{3}$$

In other words, two photons of the electromagnetic field in

 0038-5646/90/040649-04$03.00

one state and two quanta of the longitudinal field participate in four-wave mixing. The longitudinal plasma wave with the higher frequency, Ω_2, has a negative energy under the conditions

$$\Omega_2^2/\omega_0^2-1>(N_s/N)^{1/3}\gamma^{-1},\quad \Omega_2-q_2V_0<0,\quad N_s/N\ll 1. \tag{4}$$

In this case the plasma wave of positive energy has a frequency $\Omega_1=\omega_0+\delta$, where

$$\delta/\omega_0<(1/2\gamma)(N_s/N)^{1/3}\ll 1. \tag{5}$$

It follows from (2)–(5) that the frequency ω of the electromagnetic wave which is excited is given by

$$\omega\approx\omega_0(N_s/N)^{1/6}\gamma^{1/2}(\delta/\omega_0)^{-1/4}. \tag{6}$$

In other words, for a certain relationship between the parameters N_s/N and γ this frequency may be substantially higher than the plasma frequency of the plasma, ω_0. If there are random inhomogeneities in the plasma density, the wave-number matching in (3) will be disrupted, and the effect should be a random phase mismatch of the interacting waves. Let us assume that the inhomogeneities of the plasma electron density N are small, one-dimensional (the variation is along the x axis), and large in scale in comparison with the wavelengths of these electrons:

$$N(x)=N_0+\Delta N(x),\quad \langle\Delta N(x)\rangle=0,\quad |\Delta N/N_0|\ll 1. \tag{7}$$

In the case of a weak nonlinearity, the corresponding wave perturbations for the electromagnetic and plasma waves can then be sought in the form

$$u_i(x,t)=A_i(x,t)\exp\left\{i\left(\omega_it+\int q_i(x)\,dx\right)\right\}+\text{c.c.}, \tag{8}$$

where $A_i(x,t)$ are slowly varying complex amplitudes of the waves. In the linear case, expression (8) obviously corresponds to the well-known approximation in geometric optics. In this case Eq. (3) for the wave numbers becomes

$$q_2=q_1+k+k+\Delta(x), \tag{9}$$

where $\Delta(x)=\Delta q_1(x)+2\Delta k(x)-\Delta q_2(x)$, and $\langle\Delta(x)\rangle=0$.

Analysis of (2)–(5) leads to the expressions

$$\Delta q_1(x)\approx\frac{\omega_{0s}}{2V_0}\frac{\Delta N(x)}{N_0}\left(\frac{2\delta}{\omega_0}\right)^{-3/2},$$
$$\Delta q_2/\Delta q_1\approx(\omega_0/\Omega_2)^2(2\delta/\omega_0)^{3/2}\ll 1,\quad \Delta k/\Delta q_1\approx\Delta q_2/\Delta q_1\ll 1, \tag{10}$$

from which it follows that the phase mismatch of the waves is determined primarily by fluctuations in the wave number of the positive-energy longitudinal wave.

Substituting (8) and (9) into (1), using (3), and also using the asymptotic method of Ref. 3, we can derive a system of steady-state equations for the dimensionless complex amplitudes:

$$da_1/dx=-i\sigma_1a_2b^{*2}\exp\left(-i\int\Delta(x)\,dx\right)+ia_1(\alpha_1|a_1|^2+\alpha_2|a_2|^2+\alpha_3|a_3|^2),$$
$$da_2/dx=i\sigma_2a_1b^2\exp\left(i\int\Delta(x)\,dx\right)-ia_2(\beta_1|a_1|^2+\beta_2|a_2|^2+\beta_3|a_3|^2), \tag{11}$$
$$da_3/dx=-i\sigma_3a_2a_1^*b^*\exp\left(-i\int\Delta(x)\,dx\right)+ia_3(\delta_1|a_1|^2+\delta_2|a_2|^2+\delta_3|a_3|^2),$$

where $a_{1,2,3}=A_{1,2,3}(mc^2N_s\gamma)^{-1/2}$, subscripts 1 and 2 refer to the longitudinal waves, subscript 3 refers to the electromagnetic waves,

$$\sigma_1\approx\frac{\gamma^4}{4\pi}\frac{\delta\omega_{0s}}{\omega V_1},\quad \sigma_2\approx\frac{\gamma^4}{4\pi}\left(\frac{2\delta}{\omega_0}\right)^{1/2}\frac{\omega_{0s}^2}{\omega V_0},$$
$$\sigma_3\approx\frac{2\gamma^4}{\pi}\frac{\omega_{0s}^2}{\omega c},\quad \alpha_1\approx\frac{\gamma}{4\pi}\left(\frac{\delta}{\omega_0}\right)^2\frac{\omega_0}{V_1}$$
$$\alpha_2\approx\frac{\gamma}{8\pi}\frac{\delta}{\omega_0}\frac{\omega_0}{V_1},\quad \alpha_3\approx\frac{\gamma^4}{4\pi}\frac{\delta}{\omega_0}\frac{\omega_0\omega_{0s}}{\omega V_1},$$
$$\beta_1\approx\frac{\gamma^2}{8\pi}\frac{\delta}{\omega_0}\frac{\omega_{0s}}{V_0},\quad \beta_2\approx\frac{\gamma^2}{16\pi}\frac{\omega_{0s}}{V_0},$$
$$\beta_3\approx\frac{\gamma^2}{16\pi}\frac{\omega_{0s}^3}{\omega_0V_0},$$
$$\delta_1\approx\frac{\gamma^4}{\pi}\frac{\delta}{\omega_0}\frac{\omega_{0s}^2}{\omega c},\quad \delta_2\approx\frac{\gamma^4}{2\pi}\frac{\omega_{0s}^2}{\omega c},$$
$$\delta_3\approx\frac{\gamma^8}{2\pi}\frac{\omega_{0s}^4}{\omega^3c},$$

and V_1 is the group velocity of the longitudinal wave of frequency Ω_1.

2. ANALYSIS OF THE BEHAVIOR OF THE AVERAGE WAVE INTENSITIES

From (11) we can easily derive equations for the real quantities, first making the substitutions $a_{1,2}=u_{1,2}(\sigma_{1,2}\sigma_3)^{-1/2}$, $a_3=u_0(\sigma_1\sigma_2)^{-1/2}$, $x=x_\delta(\sigma_1\sigma_2)^{-1/2}$, and $u_{1,2,0}=U_{1,2,0}\exp(i\varphi_{1,2,0})$:

$$dU_{1,2}/dx=U_0^2U_{2,1}\sin\phi,\quad dU_0/dx=U_0U_1U_2\sin\phi,$$
$$\frac{d\phi}{dx}=\left(\frac{U_1U_0^2}{U_2}+\frac{U_2U_0^2}{U_1}+2U_1U_2\right)\cos\phi-h_0U_0^2-h_1U_1^2-h_2U_2^2+\Delta(x), \tag{12}$$
$$h_0=(\sigma_2/\sigma_1)^{1/2}\sigma_3^{-1}(\alpha_3+\beta_3+\delta_3),\quad h_1=(\sigma_1\sigma_2)^{-1/2}(\alpha_1+\beta_1+\delta_1),$$
$$h_2=(\sigma_1/\sigma_2)^{1/2}\sigma_3^{-1}(\alpha_2+\beta_2+\delta_2),$$

where $\phi(x)=\varphi_2-\varphi_1-2\varphi_0$ is the phase difference of the interacting waves.

Equations (12) differ from the system of equations given in Ref. 2 in the presence of the random quantity $\Delta(x)$ in the last equation. The presence of this term is of fundamental importance, since it is this term which causes the phase difference $\phi(x)$ and the amplitudes $U_{1,2,0}(x)$ to be stochastic quantities. We should accordingly analyze any moments of the field below. Clearly one of the most important characteristics is the average intensity of the interacting waves [the average here is over the ensemble of inhomogeneities $\Delta(x)$]. We accordingly introduce the quantities

$$J_k=U_k^2\quad(k=0,1,2), \tag{13}$$

which are proportional to the intensities. The equations for these new quantities take the form

$$\frac{dJ_{1,2,0}}{dx}=h(J)\sin\phi,$$
$$\frac{d\phi}{dx}=\sum_{k=0}^{2}\left(\frac{\partial h}{\partial J_k}\cos\phi-h_kJ_k\right)+\Delta(x),$$
$$h(J)=2(J_0^2J_1J_2)^{1/2}. \quad (14)$$

It follows from (14) that the first three equations have integrals:

$$J_1-J_0=C_1,\ J_2-J_0=C_2,\ C_{1,2}=\text{const}. \quad (15)$$

In other words, the wave intensities differ by only a constant.

We turn now to a particular case, but one of fundamental importance: that in which the wave intensities at the entrance to the nonlinear slab ($x=0$) are equal, $J_1(0)=J_2(0)=J_0(0)$. It follows from (15) that under this initial condition the wave intensities become equal at any point in the slab:

$$J_1(x)=J_2(x)=J_0(x)=J(x). \quad (16)$$

Equations (14) then simplify, becoming

$$dJ/dx=2J^2\sin\phi,$$
$$d\phi/dx=4J(\cos\phi-\alpha/4)+\Delta(x), \quad (17)$$
$$\alpha=h_0+h_1+h_2.$$

It can be seen from (17) that under the condition $\Delta(x)=0$ we have the integral

$$\Gamma=J^2(\alpha/4-\cos\phi)=\text{const}. \quad (18)$$

The solution for $J(x)$ is therefore bounded and furthermore periodic under the condition[1)] $\alpha/4>1$.

Figure 1 shows J^2 versus the polar angle ϕ. We see that the function $J^2(\phi)$ is asymmetric. It is shown below that this asymmetry gives rise in the case $\Delta(x)\neq 0$ to diffusive displacement of the phase trajectories, which in turn causes exponential growth of the average intensity. The spatial period is given by

$$\Lambda={}^1/_4\Gamma^{-1/2}\int_0^{2\pi}(\alpha/4-\cos\phi)^{-1/2}\,d\phi. \quad (19)$$

Let us now assume that there is a random phase deviation $[\Delta(x)\neq 0]$. In this case the quantity Γ is not a constant. Along with ϕ, it is a realization of a two-dimensional Markov random diffusion process (Γ,ϕ). We assume that $\Delta(x)$ is a δ-correlated Gaussian random process with a correlation function

$$\langle\Delta(x)\Delta(x_1)\rangle=\sigma^2\delta(x-x_1) \quad (20)$$

($\langle\ldots\rangle$ means a statistical average). The two-dimensional diffusion process (Γ,ϕ) then satisfies the Itô stochastic equations[4]

$$d\Gamma=\frac{\sigma^2}{2}\Gamma\frac{\cos\phi}{\alpha/4-\cos\phi}dx+\Gamma\sigma\frac{\sin\phi}{\alpha/4-\cos\phi}dW_x,$$
$$d\phi=4\Gamma^{1/2}(\alpha/4-\cos\phi)^{1/2}dx+\sigma dW_x, \quad (21)$$

where

$$\Delta W_x=\sigma^{-1}\int_0^{\Delta x}\Delta(x)\,dx$$

realizes a Wiener random process: $\langle\Delta W_x\rangle=0$, $\langle(\Delta W_x)^2\rangle=\Delta x$.

It is difficult to analyze (21) in its general form, so we turn to the limiting case in which the random shift of the phase ϕ over one oscillation period Λ is small[2)]:

$$\langle(\Delta\phi)^2\rangle\approx\sigma^2\Lambda\ll 1. \quad (22)$$

In this case we can ignore the second (fluctuating) term in the second equation of system (21), and we can assume that the variation of the phase ϕ over one period is deterministic and is described by the equation

$$d\phi/dx=4\Gamma^{1/2}(\alpha/4-\cos\phi)^{1/2}, \quad (23)$$

while the quantity $\Gamma(x)$ is a slowly varying random process which is described by the first linear equation in (21), whose solution is known:[4]

$$\Gamma(x)=\Gamma_0\exp\left\{\frac{\sigma^2}{2}\int_0^x\left[\frac{\cos\phi(\zeta)}{\alpha/4-\cos\phi(\zeta)}-\frac{\sin^2\phi(\zeta)}{(\alpha/4-\cos\phi(\zeta))^2}\right]d\zeta+\sigma\int_0^x\frac{\sin\phi(\zeta)}{\alpha/4-\cos\phi(\zeta)}dW_\zeta\right\}. \quad (24)$$

The quantity $\phi(\xi)$ in (24) is the solution of Eq. (23) with $\Gamma=\Gamma_0$. Taking the average of (24) by the well-knolwn Itô procedure, and setting $x=\Lambda$, we find

$$\langle\Gamma(\Lambda)\rangle=\Gamma_0\exp\left\{\frac{\sigma^2}{2}\int_0^\Lambda\frac{\cos\phi(\zeta)}{\alpha/4-\cos\phi(\zeta)}d\zeta\right\}=\Gamma_0\exp\{\delta\Lambda\},$$
$$\delta=\frac{\sigma^2}{2\Lambda}\int_0^\Lambda\frac{\cos\phi(\zeta)}{\alpha/4-\cos\phi(\zeta)}d\zeta=\frac{\sigma^2}{2}\frac{\Lambda_1}{\Lambda}, \quad (25)$$
$$\Lambda_1=\frac{1}{4\Gamma_0^{1/2}}\int_0^{2\pi}\frac{\cos\phi\,d\phi}{(\alpha/4-\cos\phi)^{3/2}}>0.$$

It follows from (25) that over a distance Λ the average value of $\langle\Gamma(\Lambda)\rangle$ increases by a factor of $\exp\{\delta\Lambda\}$ with respect to its initial value Γ_0 and that the quantity δ can be interpreted as the average drift of process Γ over a distance equal to the period Λ in the unperturbed problem.

Strictly speaking, expressions (24)–(25) apply only at small distances ($\sigma^2x\ll 1$). For large distances we can use a

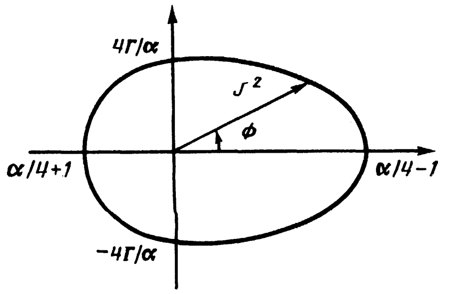

FIG. 1. Intensity squared versus the polar angle ϕ.

cruder method involving an average over the period Λ. This cruder method can be summarized as follows: Since there is only a negligible diffusion of the phase ϕ over a distance on the order of the period Λ, the drift and diffusion coefficients in the first equation of Eqs. (21) are replaced by their average values over the period under the assumption that over the distance Λ the angle ϕ varies in a deterministic way. The average value of the drift coefficient was found above:

$$\bar{a}_\Lambda=\delta=(\sigma^2/2)\Lambda_1/\Lambda.$$

The average value of the diffusion coefficient is found from

$$\bar{D}_\Lambda^2=\frac{\sigma^2}{\Lambda}\int_0^\Lambda\frac{\sin^2\phi(\zeta)}{(\alpha/4-\cos\phi(\zeta))^2}d\zeta$$
$$=\frac{\sigma^2}{4\Gamma_0^{1/2}}\int_0^{2\pi}\frac{\sin^2\phi\,d\phi}{(\alpha/4-\cos\phi)^{3/2}}=\frac{2}{3}\sigma^2\frac{\Lambda_1}{\Lambda}. \tag{26}$$

The "average" equation for $\Gamma(x)$ becomes

$$d\Gamma=\bar{a}_\Lambda\Gamma dx+\bar{D}_\Lambda\Gamma dW_x. \tag{27}$$

We write its solution in the following form, in accordance with Ref. 4:

$$\Gamma=\Gamma_0\exp\{(\bar{a}_\Lambda-\bar{D}_\Lambda^2/2)x+\bar{D}_\Lambda dW_x\}. \tag{28}$$

Using (28), we can find all the moments of the random quantity $\Gamma(x)$. For example, the mean value $\langle\Gamma(x)\rangle$ is, as above,

$$\langle\Gamma(x)\rangle=\Gamma_0\exp\left(\frac{\sigma^2}{2}\frac{\Lambda_1}{\Lambda}x\right). \tag{29}$$

Consequently, the existence of random inhomogeneities in a medium, which leads to a stochastic phase mismatch of the interacting waves, causes an exponential growth of the quantity $\langle\Gamma(x)\rangle$ and thus of the average intensity [see (18)]. A more general assertion can be made on the basis of this analysis: In any physical system in which an "explosion" stabilizes in such a way that intensities vary periodically in space (or time), a stochastic instability will occur because of the random phase mismatch of waves.

We are indebted to B. S. Abramovich and V. Yu. Trakhtengerts for discussions.

[1)]There is also a periodic solution in the case of three-wave mixing (with a nonlinear frequency shift) if the initial wave amplitudes are different.[1]
[2)]The replacement $\phi\to-\phi$ was made in (21) so that an increase in x would correspond to an increase in ϕ.

[1]H. Wilhelmsson and J. C. Weiland, *Coherent Nonlinear Interaction of Waves in Plasmas*, Pergamon, New York, 1977 (Russ. Transl. Energoizdat, Moscow, 1981, p. 90).
[2]S. M. Faĭnshteĭn, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **21**, 754 (1978).
[3]V. N. Tsytovich, *Nonlinear Effects in Plasma*, Nauka, Moscow, 1967; Plenum, New York, 1970.
[4]R. Z. Khas'minskiĭ, *Stability of Systems of Differential Equations with Random Perturbations in Their Parameters*, Nauka, Moscow, 1969, p. 132.

Translated by Dave Parsons