Thermal self-focusing of sawtooth waves

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Thermal self-focusing and defocusing of sawtooth waves are investigated. The geometric approximation of acoustics and the assumption that there are no aberrations of the self-induced lens are employed. The characteristic temporal and spatial scales of the process, the wave amplitude, and beam width are calculated. The specific nature of the effect, as compared with the familiar thermal self-focusing of quasi-harmonic waves, is noted.

As is well known, beams of intense waves can undergo self-focusing (defocusing). This effect, already predicted in 1962 for light,[1] and later for sound,[2] has been studied in detail in nonlinear optics. The experimental observation of thermal self-focusing (TSF) of acoustic waves took place relatively recently.[3,4] The phenomenon of TSF comes about because of the dependence of the velocity of the wave on the temperature and of the nonuniform heating of the medium by the beam. At the present time, TSF has been well analyzed for quasi-harmonic sound waves,[5,6] where the analogy with optical TSF has been used effectively.[7] According to such an approach, the TSF is reduced upon a decrease in the coefficient of linear absorption of sound α, and it disappears as $\alpha \to 0$. The specifics of acoustics are such that this conclusion is not always valid.

Actually, because of the absence of dispersion, the shape of intense sound waves undergoes strong nonlinear distortion and, as the wave propagates, "discontinuities" are formed—weak shock waves. In this case, even if the absorption coefficient α is small, an effective nonlinear dissipation of the energy of the wave takes place, and the medium is heated, i.e., thermal self-focusing occurs. The corresponding change in the sound velocity has been observed experimentally.[8,9] Therefore, the acoustic beams should undergo self-focusing even in a medium with small α.

An experiment has been described in Ref. 10 on observation of TSF due to nonlinear absorption of ultrasonic waves. A beam of 20-W power, width 30 mm and frequency 2 MHz propagates through acetone, which has a small linear absorption coefficient. It turned out that the intensity of the wave on the axis of the beam increased significantly (by a factor of about 1.5), which was not explained by linear sound absorption, and was due to the effect of nonlinear quadratic effects. We note that the role of nonlinear absorption increases with increase in the amplitude of the wave, so that linear absorption can prove to be unimportant in the TSF of powerful nondispersive waves.

It is well known that the shape of a powerful sound wave transforms from sinusoidal into sawtooth as it propagates. In this case, the absorption of the wave becomes purely nonlinear and does not depend on the viscosity and thermal conductivity of the medium. The task of the present article is the description of thermal self-focusing of such waves. We note that most of the features of TSF of sound are intrinsic to the self-focusing of nondispersive waves of any sort, for example, ion-acoustic and magneto-acoustic waves in a plasma, or waves in particle flows. Therefore, a number of results of the developed theory extend beyond the framework of problems of acoustics and are applicable to a broad class of nonlinear problems.

1. DERIVATION OF THE EQUATIONS

In the propagation of acoustic waves in ordinary liquids, three modes of excitation arise: acoustic, entropic and hydrodynamic. The effect of TSF is the result of the interaction of the first two modes. However, hydrodynamic modes can also play a large role in TSF: the development of "acoustic wind" leads to defocusing;[5] the "wind" and convective flows agitate the liquid strongly, which weakens the TSF.[11,12] It is assumed below that the hydrodynamic mode is not excited (the medium is at rest.

For a description of the acoustic beam, we used a modified equation of the Khokhlov–Zabolotskaya–Kuznetsov type[5,12]

$$\frac{\partial}{\partial \tau}\left[\frac{\partial p}{\partial x}-\frac{\delta T}{c_0}\frac{\partial p}{\partial \tau}-\frac{\varepsilon}{\rho_0 c_0^3}p\frac{\partial p}{\partial \tau}-\frac{b}{2\rho_0 c_0^3}\frac{\partial^2 p}{\partial \tau^2}\right]=\frac{c_0}{2}\Delta_\perp p, \quad (1)$$

where p is the acoustic pressure, T the temperature of the medium averaged over the period of the wave, $\delta = c_0^{-1}(\partial c/\partial T)_p$ is the temperature coefficient of the sound velocity c, c_0, ρ_0 are the unperturbed sound velocity and density of the medium, ε, b are the parameters of nonlinearity and dissipation,[13] x is the coordinate along the axis of the beam, $\tau = t - x/c_0$, t is the time, $\Delta_\perp = r^{-1}\partial/\partial r(r\partial/\partial r)$ is Laplace's operator with respect to the transverse coordinate r.

We shall calculate the temperature field with the help of the inhomogeneous equation of heat conduction

$$\frac{\partial T}{\partial t}-\frac{\varkappa}{\rho_0 C_p}\Delta_\perp T=\frac{b}{c_0^4\rho_0^3 C_p}\overline{\left(\frac{\partial p}{\partial \tau}\right)^2}. \quad (2)$$

The expression on the right-hand side, where the superior bar indicates averaging over the period of the acoustic wave, describes the transformation of the acoustic energy into thermal energy via absorption. In Eq. (2) $\varkappa$ and C_p are the coefficient of thermal conductivity and the heat capacity of the system, respectively.

The set of equations (1), along with the initial and boundary conditions, can be used in the calculation of the interaction of acoustic and entropy modes in the sound beam. However, Eq. (1) can only be solved numerically, even upon neglect of the thermal self-action ($\delta \to 0$). In what follows, its simplification is possible in situations when the acoustic wavelength λ is much less than the characteristic scale of the temperature inhomogeneity L. For this purpose,

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we set

$$p=p(x,\,r,\,\theta=\tau-\psi(x,\,r)/c_0)$$

in (1). Here $\psi(x,r)$ is the shift in the wave front because of heating of the medium. Transforming to the approximation of geometric acoustics $\lambda/L\to 0$, we get from (1)

$$\frac{\partial p}{\partial x}-\frac{\varepsilon}{\rho_0 c_0^3}p\frac{\partial p}{\partial \theta}-\frac{b}{2\rho_0 c_0^3}\frac{\partial^2 p}{\partial \theta^2}+\frac{\partial \psi}{\partial r}\frac{\partial p}{\partial r}+\frac{\Delta_\perp \psi}{2}p=0, \quad (3)$$

$$\frac{\partial \psi}{\partial x}+{}^1/_2\left(\frac{\partial \psi}{\partial r}\right)^2+\delta T=0. \quad (4)$$

Equation (3) has the structure of the Burgers equation, which is used for the description of plane nonlinear waves,[13] but differs from it by the the last two terms, which take into account the change in the cross section of the ray tubes. Equation (4) is the eikonal equation, which characterizes the curvature of the rays because of the increase in the temperature of the medium T.

We note that diffraction must be insignificant over the scale of self-focusing x_f for the possibility of application of the approximation of geometric acoustics. The diffraction length x_d can be estimated at $x_d\sim\pi a^2/\lambda$, where a, λ are the characteristic transverse dimensions of the beam and the wavelength. As will be seen from what follows, the length x_f decreases rapidly with increase in the wave amplitude. Therfore, $x_f<x_d$ for sufficiently powerful beams, i.e., the diffraction can be neglected.

Let us specify the form of the wave. If, for example, we represent it as harmonic, i.e.,

$$p=A(x,\,r)\sin\omega\theta,$$

then we get from (3) the same equation for the amplitude A as in the description of the optical TSF.[7] We are interested in sawtooth waves. The harmonic wave becomes sawtooth over a distance

$$x_p\sim\rho_0 c_0^3/(\varepsilon\omega A_0),$$

where A_0 is the wave amplitude at the entrance to the medium. In propagation of megahertz acoustic waves with amplitudes of several atmospheres, x_p amounts to several centimeters. Therefore, the sawtooth waves are the usual ones for experiment in the region of nonlinear acoustics. For the description of such waves, we use the representation of Khokhlov for the shape of the wave over a single period:

$$p(x,r,\theta)=A(x,r)\left\{\mathrm{th}\left[\frac{\varepsilon A(x,r)\theta}{b}\right]-\frac{\omega\theta}{\pi}\right\},$$

$$-\frac{\pi}{\omega}<\theta<\frac{\pi}{\omega}. \quad (5)$$

Here A is the wave amplitude and, $2\pi/\omega$ is its period. The expression (5) is the exact solution of the Burgers equation[13] and as $b\to 0$ it takes on a sawtooth profile. The limit $b\to 0$ corrsponds to the approximation of large values of the acoustic Reynolds number ($b\ll 2\pi\varepsilon A/\omega$), which is typical for powerful sound waves. Equations (3) and (2) here yield

$$\frac{\partial A}{\partial x}+\frac{\varepsilon\omega}{\pi\rho_0 c_0^3}A^2+\frac{\partial \psi}{\partial r}\frac{\partial A}{\partial r}+\frac{\Delta_\perp \psi}{2}A=0, \quad (6)$$

$$\frac{\partial T}{\partial t}-\frac{\varkappa}{\rho_0 C_p}\Delta_\perp T=\frac{2\varepsilon\omega}{3\pi c_0^4\rho_0^3 C_p}A^3. \quad (7)$$

The resultant set of equations (6), (7), and (4) describe the thermal self-action of sawtooth waves. It is seen from (6) that the absorption of the wave is purely nonlinear; in the linear case, in place of the second term we would have had αA. The form of the right-hand side of (7) is also connected with this feature: the heat release power is proportional to the cube of the amplitude (in the linear case it is proportional to the square).

A substantial simplification of Eqs. (4) and (6) for the amplitude and phase of the waves is possible on the basis of the aberration-free approximation.[7] We assume that the wave front is always spherical, with only its curvature β changing:

$$\psi(x,\,r,\,t)=\varphi(x,\,t)+r^2\beta(x,\,t)/2,$$

where φ is the shift of the front on the axis of the beam. This assumption is validated in the case a parabolic transverse distribution of temperature T. Substituting ψ in this form in (6), we get

$$\frac{\partial A}{\partial x}+\frac{\varepsilon\omega}{\pi\rho_0 c_0^3}A^2+\beta\frac{\partial}{\partial r}(rA)=0. \quad (8)$$

Then the amplitude A can be expressed exactly in terms of the curvature of the wavefront β. Actually, introducing the auxiliary function

$$f(x,t)=\exp\left[\int_0^x \beta(x',t)\,dx'\right],$$

we find from (8)

$$\frac{A}{p_0}=\frac{1}{f}\Phi\left(\frac{r}{af}\right)\left[1+x_p^{-1}\Phi\left(\frac{r}{af}\right)\int_0^x\frac{dx'}{f(x',t)}\right]^{-1}. \quad (9)$$

Here $p_0=A(x=0,r=0)$ is the wave amplitude on the axis of the beam at the entrance to the medium. The function $\Phi(\xi)$ describes the transverse distribution of the wave amplitude at the entrance: $A(x=0,r)/p_0=\Phi(r/a)$, for example, for Gaussian beams, $\Phi(\xi)=\exp(-\xi^2)$; a is the initial radius of the beam, $x_p=\pi\rho_0 c_0^3/(\varepsilon\omega p_0)$ is the scale of nonlinear absorption of the wave. With application of Eq. (9), the problem of finding the structure of $A(x,r,t)$ is considerably simplified, since it is reduced to finding a function of two variables, $f(x,t)$. As follows from (4), f is described by the equation

$$f^{-1}\partial^2 f/\partial x^2=\delta T_2, \quad (10)$$

where $T_2(x,t)$ is the coefficient of r^2 in the expansion of the temperature T in the transverse coordinate: $T=T_0-T_2r^2/2+\ldots$ We denote by $t_0=\rho_0 C_p a^2/(12\varkappa)$ the characteristic time of establishment of the temperature. Generally speaking, for calculation of T_2 we must solve (7) relative to T. However, there are two cases of practical importance in which it is not necessary to find T in order to determine T_2: 1) for $t\gg t_0$ (stationary TSF) and 2) for $t\ll t_0$ (initial stage of TSF). We now consider these two cases.

2. STATIONARY TSF

The established regime corresponds to the condition $\partial T/\partial t=0$ in Eq. (7). Expanding T in a series in the transverse coordinate r and taking (9) into account, we obtain an expression for T_2:

$$T_2 = \frac{\varepsilon\omega p_0^3}{3\pi\varkappa c_0^4 \rho_0^2} f^{-3} \left[1 + x_p^{-1} \int_0^x \frac{dx'}{f(x')} \right]^{-3}.$$

From this and from (10) follows an equation for $f(x)$. For convenience we normalize the longitudinal coordinate: $z = x/x_0$, where

$$x_0 = \pi^2 |\delta| \rho_0 c_0^5 / (3\varkappa\varepsilon^2\omega^2)$$

is the characteristic length, which does not depend on the amplitude of the wave p_0. With account of the boundary conditions, the following problem arises for finding the function $f(x)$:

$$\frac{d^2 f}{dz^2} = \frac{\Pi^3}{f^2} \operatorname{sgn}(\delta) \left[1 + \Pi \int_0^z \frac{dz'}{f} \right]^{-3}, \tag{11}$$

$$f|_{z=0} = 1, \quad \left. \frac{df}{dz} \right|_{z=0} = K \tag{12}$$

Here $\Pi = x_0/x_p = p_0/P$ is the dimensionless wave amplitude at the entrance to the medium,

$$P = 3\varkappa\varepsilon\omega / (\pi|\delta| c_0^2)$$

is the characteristic pressure, $K = x_0/R$ is the dimensionless curvature of the wavefront at the entrance to the medium, $R = \beta^{-1}(x = 0)$ is the radius of curvature. Thus, finding the amplitude $A(x,r)$ of the wave in the case of stationary *TSF* reduces to the calculation of f from (11) and (12) and use of the functional relation (9).

The solution of the problem (11), (12) depends on two parameters—the dimensionless amplitude Π and the curvature of the wavefront K. Moreover, it follows from (11) that the behavior of the function $f(z)$ [and, it would appear, $A(x,r)$ also] changes qualitatively upon change in the sign of δ. In the case $\delta > 0$, defocusing takes place, for $\delta < 0$, focusing.

Figure 1 shows the results of a numerical calculation of the function $A(x,r)$ from Eqs. (9)–(12) in a medium with $\delta < 0$ at $\Pi = 0.1$ and 1. It has been assumed that the beam is Gaussian at the entrance to the medium, i.e.,

$$\Phi(r/a) = \exp(-r^2/a^2),$$

with a plane wavefront ($K = 0$). It is seen that with increase in the distance z the amplitude A initially decreases and then increases, and a focus is formed at a certain distance, where $A \to \infty$.

Near the focus, the analysis that has been carried out is incorrect, since the initial approximations of geometric acoustics and a sawtooth shape of the profile are invalid. It is seen from a comparison of Figs. 1a and 1b how the structure of the beam changes with increase in the amplitude Π. At $\Pi = 0.1$, the amplitude of the wave falls off significantly because of the nonlinear absorption as it progresses into the medium, in spite of the self-focusing (Fig. 1a). As a consequence of the nonlinear character of the absorption, the transverse profile of the beam changes from Gaussian into a more uniform one (the effect of isotropization of the beam[13]). At large amplitudes of the wave ($\Pi = 1$ in Fig. 1b) the nonlinear absorption and the isotropization are not present to any extent, since the focus is formed at smaller distances.

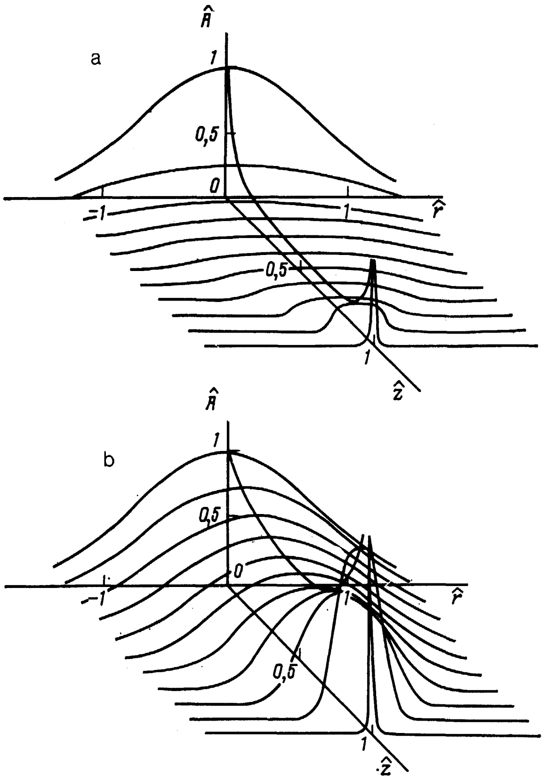

FIG. 1. Distribution of wave amplitude in self-focusing ($\hat{A} = A/p_0$, $\hat{z} = z/z_f$, $r = r/a$): *a*—$\Pi = 0.1$, $z_f = 203$; *b*—$\Pi = 1$, $z_f = 2.36$.

We shall further characterize the beam by the amplitude of the wave on the axis A_0 and the transverse radius of the beam a_0 at the level e^{-1}:

$$A(r = a_0, x) = A(r = 0, x)/e.$$

It follows from Eq. (9) that

$$\frac{A_0(z)}{p_0} = f^{-1} \left[1 + \Pi \int_0^z \frac{dz'}{f} \right]^{-1}, \tag{13}$$

$$\frac{a_0(z)}{a} = f \left\{ 1 + \ln \left[1 + (1 - e^{-1}) \Pi \int_0^z \frac{dz'}{f} \right] \right\}^{1/2}. \tag{14}$$

Figure 2 shows these dependences for different Π at $K = 0$, $\delta < 0$. The effects of nonlinear absorption and isotropization are clearly seen, especially for small Π: A_0 decreases initially while a_0 increases. We note that the length of self-focusing z_f decreases rapidly with increase in the amplitude of the wave Π. This is clearly seen from the Table, in which the results of calculation of the value of z_f are given for different values of Π.

The asymptote of the function $f(z)$ can be obtained from (11) in the case of large and small amplitudes of the wave Π. As is seen from Fig. 2, the nonlinear absorption becomes weaker and weaker with increase in Π. This means that the integral on the right-hand side of (11) plays an ever decreasing role. If we neglect it, the we obtain the equation

$$\frac{d^2 f}{dz^2} = \operatorname{sgn}(\delta) \frac{\Pi^3}{f^2},$$

which has an analytic solution. If the defocusing medium ($\delta > 0$), this solution can be described in the form

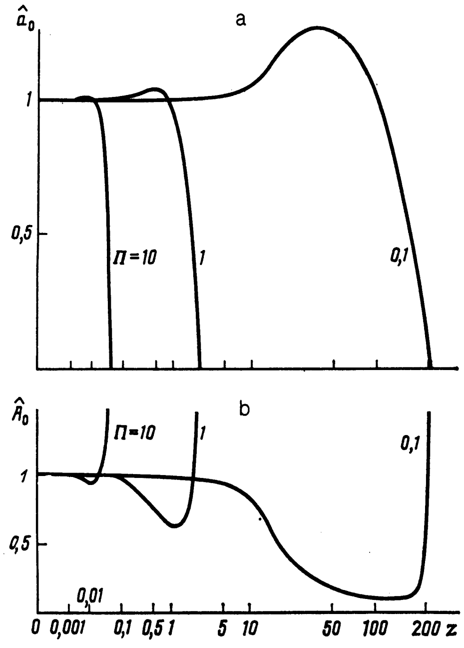

FIG. 2. Dependence of the beam characteristics on the distance z for various wave amplitudes Π: a—$\hat{a}_0 = a_0/a$ is the dimensionless radius of the beam; b—$A = A_0/p_0$ is the dimensionless amplitude of the wave on the axis.

$$\pm(2\Pi^3)^{1/2}z = {}^1/_2\gamma^{-3/2}\{[(2\gamma f-1)^2-1]^{1/2}-[(2\gamma-1)^2-1]^{1/2} +\ln[2\gamma f-1+((2\gamma f-1)^2-1)^{1/2}]-\ln[2\gamma-1+((2\gamma-1)^2-1)^{1/2}]\},$$

where $\gamma = K^2/(2\Pi^3) + 1$. In the focusing region ($\delta < 0$) the form of the solution is obtained differently, depending on the sign of the quantity $\Gamma = K^2/(2\Pi^3) - 1$:

$$\pm(2\Pi^3)^{1/2}z = {}^1/_2\Gamma^{-3/2}\{[2\Gamma f+1)^2-1]^{1/2}-[(2\Gamma+1)^2-1]^{1/2} -\ln[2\Gamma f+1+((2\Gamma f+1)^2-1)^{1/2}]+\ln[2\Gamma+1+((2\Gamma+1)^2-1)^{1/2}]\} \quad \text{for } \Gamma>0,$$

$$\pm z = {}^2/_3(f^{3/2}-1) \quad \text{for } \Gamma=0,$$

$$\pm(2\Pi^3)^{1/2}z = {}^1/_2|\Gamma|^{-3/2}\{[1-(2\Gamma f+1)^2]^{1/2}-[1-(2\Gamma+1)^2]^{1/2} + \arcsin(2\Gamma f+1)-\arcsin(2\Gamma+1)\}, \quad \text{for } \Gamma<0.$$

It follows from the last expression that a beam with a plane wavefront ($K = 0, \Gamma = -1$) "collapses" at a distance $z_f^* = \pi(2\Pi)^{-3/2}$. The Table gives the values of z_f^* for different Π. It is seen that the approximation to the exact focus is quite satisfactory for $\Pi \gtrsim 5$.

Upon decrease in Π, the focal distance increases. The fact that in this case the effect of nonlinear absorption increases (Fig. 2) means that at small Π the energy of the wave is essentially dissipated at distances that are much less than the focal distance. This allows us to apply the "thin lens" approximation. We note that the quantity $r_0 f$ is the distance from the axis of the beam to the acoustic ray passing through the circumference ($x = 0$, $r = r_0$). In the considered approximation, in the region of strong acoustic absorption, where bending of the rays also occurs, the distance from these rays to the axis decreases insignificantly. Setting $f \approx 1$ in (11) in correspondence with this approximation, we obtain

$$\frac{d^2f}{dz^2} \approx \operatorname{sgn}(\delta)\frac{\Pi^3}{(1+\Pi z)^3}.$$

Then, with account of (12),

$$df/dz = K + \operatorname{sgn}(\delta)(\Pi^2/2)[1-(1+\Pi z)^{-2}].$$

After passage by the wave of the region of strong acoustic absorption, when $\Pi z = x/x_p \gg 1$, we have

$$f \approx 1, \quad df/dz \approx K + \operatorname{sgn}(\delta)\Pi^2/2, \quad d^2f/dz^2 \approx 0.$$

Neglecting the thickness of the lens, we obtain

$$f \approx 1 + z[K + \operatorname{sgn}(\delta)\Pi^2/2].$$

At $K = 0$ and $\delta < 0$ we have for the focal distance $z_f^{**} = 2\Pi^{-2}$. In the Table, we give the values of z_f^{**} for different Π. From a comparison with the results of numerical calculation of the quantity z_f it is seen that the "thin lens" approximation is perfectly valid at $\Pi \lesssim 2$.

There is practical interest in the effect of self-focusing on the focusing of a beam. The thermal self-action in this case hinders the focusing. Figure 3 shows the results of a calculation of the radius of the beam $a_0(z)$ at $\Pi = 10$, $\delta > 0$ for different values of the initial curvature of the wavefront K. It is seen that even for strong focusing, a nonlinear constriction of finite dimensions takes place. We recall that in the absence of self-action the radius of the constriction a_c is equal to zero by virtue of the neglect of diffraction. The quantity a_c can easily be estimated upon neglect of the nonlinear absorption: $a_c/a \approx \gamma^{-1}$. As $\Pi \to 0$, we obtain $a_c = 0$, as it should be in the linear case. Upon increase in the amplitude of the wave Π the quantity a_c increases, i.e., the quality of the focusing deteriorates because of the self-action. Moreover, the constriction moves away from the point of linear focus:

$$z_c = (8\Pi^3\gamma^3)^{-1/2}\{[(2\gamma-1)^2-1]^{1/2} + \ln[2\gamma-1+((2\gamma-1)^2-1)^{1/2}]\}.$$

We note that the effects mentioned do not require extremely

TABLE I.

Π	z_f	z_f^*	z_f^{**}
0,1	$2{,}03\cdot10^2$	–	$2{,}0\cdot10^2$
0,2	$5{,}17\cdot10$	–	$5{,}0\cdot10$
0,5	8,7	–	8,0
1	2,36	1,11	2,0
2	$6{,}7\cdot10^{-1}$	$3{,}9\cdot10^{-1}$	$5{,}0\cdot10^{-1}$
5	$1{,}39\cdot10^{-1}$	$0{,}99\cdot10^{-1}$	$0{,}8\cdot10^{-1}$
10	$4{,}46\cdot10^{-2}$	$3{,}51\cdot10^{-2}$	–
20	$1{,}47\cdot10^{-2}$	$1{,}24\cdot10^{-2}$	–
50	$3{,}52\cdot10^{-3}$	$3{,}14\cdot10^{-3}$	–
100	$1{,}21\cdot10^{-3}$	$1{,}11\cdot10^{-3}$	–

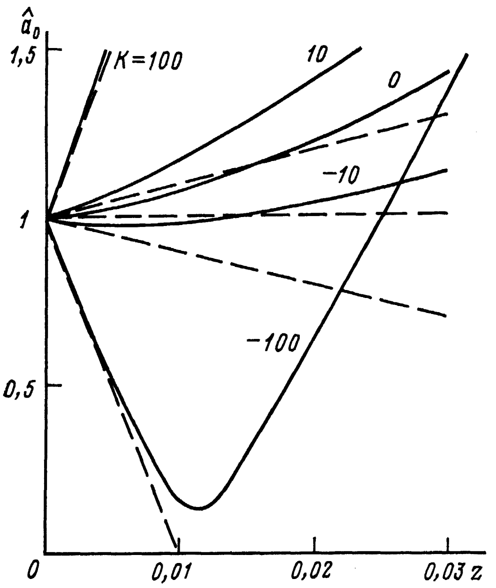

FIG. 3. Dependence of the dimensionless radius of the beam $\hat{a}_0 = a_0/a$ on the coordinate z at $\Pi = 10$ and for various curvatures of the wavefront K. The medium is defocusing: $\delta > 0$. The dashed lines are the corresponding dependences in the absence of self-action ($\Pi \to 0$).

large values of the wave amplitude. As an example, let a focused beam of sawtooth waves be created in water, and measures taken to prevent the onset of flow. We use the following beam parameters: amplitude of the wave $p_0 = 1.3$ atm frequency $\omega/2\pi = 4$ MHz, radius of the beam $a = 3$ cm, radius of curvature of the wave front $R = -9.4$ cm. In this case, calculation gives $\Pi \approx 10$, $K \approx -100$, which corresponds to the parameters of the lower curve in Fig. 3. Consequently, the constriction moves relative to the geometric focus through a distance $\Delta x = 0.15R \approx 1.4$ cm, while the radius of the constriction amounts to $a_c \approx 0.12a \approx 3.6$. In the absence of self-action and with account of diffraction, $a_c \approx Rc_0/\pi af \approx 0.36$ mm, i.e., the self-defocusing broadens the constriction, increasing a_c by an order of magnitude.

Equation (11) has the exact solution $f = \exp(-z)$, corresponding to the case $K = -1$, $\Pi = 1$. The constriction is then at infinity and the amplitude of the wave on the axis does not change: $A_0(z)/p_0 \equiv 1$. This exact solution has been used as a test in carrying out the numerical calculation of Eq. (11) on the computer.

3. INITIAL STAGE OF TSF

For some time after the onset of radiation, the processes of thermal conductivity are unimportant. We can discard the second term on the left-hand side of Eq. (7):

$$\frac{\partial T}{\partial t} \approx \frac{2\varepsilon\omega A^3}{3\pi c_0^4 \rho_0^3 C_p} \qquad (15)$$

In order to find T_2 for Eq. (10), we expand the terms of Eq. (15) in a series in the transverse coordinate r and separate the components $\propto r^2$. Here we use (9), assuming that $d^2\Phi/d\xi^2|_{\xi=0} = -2$; this latter condition can be satisfied by an appropriate choice of the quantity a—for example, this is the case for a Gaussian beam, $\Phi(\xi = r/a) = \exp(-\xi^2)$. After elementary calculations, we obtain a closed equation relative to $f(x,t)$. With account of the boundary and initial

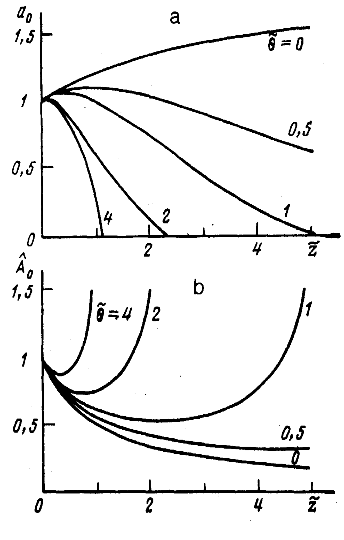

FIG. 4. Dependence of the beam charcteristics on the distance z at successive instants of time $\tilde{\theta}$ for the case $\tilde{K} = 0$, $\delta < 0$; a—$\hat{a}_0 = a_0/a$ is the dimensionless beam radius. b—$A_0 = A_0/p_0$ is the dimensionless amplitude of the wave on the axis.

conditions, we obtain

$$\frac{\partial}{\partial\theta}\left(\frac{1}{f}\frac{\partial^2 f}{\partial z^2}\right) = \frac{\Pi^3 \operatorname{sgn}(\delta)}{f^5}\left(1 + \Pi\int_0^z \frac{dz'}{f}\right)^{-4}, \qquad (16)$$

$$f|_{z=0} = f|_{\theta=0} = 1, \quad \partial f/\partial z|_{z=0} = K. \qquad (17)$$

Here we mean by $\theta = t/t_0$ the current time, normalized to the charcteristic time of thermal conductivity t_0, and for the rest we use the same notation as in the relations (11) and (12). From the sense of the approximation considered, we assume that $\theta \lesssim 1$.

It is, however, more convenient to carry out the calculation in another normalization, which depends on the amplitude p_0 of the wave, of the time and coordinate, $\tilde{z} = \Pi z = x/x_p$, $\tilde{\theta} = \Pi\theta = t/\tilde{t}_0$, where

$$\tilde{t}_0 = \varepsilon\omega\rho_0 C_p a^2/(4\pi|\delta|c_0^2 p_0).$$

Here we remove the parameter Π in Eq. (16), and the problem then becomes a single-parameter one with the parameter $\tilde{K} = K/\Pi = x_p/R$, which has the meaning of a dimensionless curvature of the wavefront. After calculation of the function f, the amplitude distribution in the beam $A(x,r,t)$ is determined from Eq. (9). The transverse radius of the beam, $a_0(\tilde{z})$, and the amplitude of the wave on the axis $A_0(\tilde{z})$ are

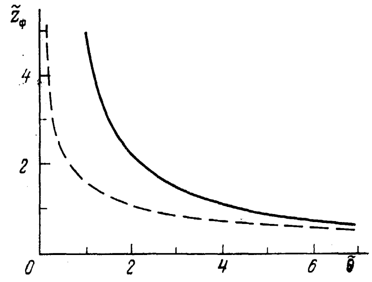

FIG. 5. Coordinate of the point of "collapse" $\tilde{z}_f$ as a function of the time $\tilde{\theta}$. The dashed curve is the corresponding $z_f^*(\tilde{\theta})$ dependence from the solution of the approximate self-similar equation.

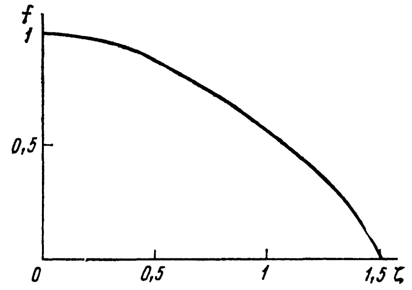

FIG. 6. Self-similar solution $f=\varphi(\zeta)$ for the case $\delta<0$.

found from formulas (13) and (14). Figure 4 shows the results of the numerical calculation of the $a_0(\tilde{z})$ and $A(\tilde{z})$ dependences for successive moments of time $\tilde{\theta}$ in a medium with $\delta<0$. The wavefront at the entrance to the medium is assumed to be planar ($\tilde{K}=0$). It is seen that, at the initial instant of time, the beam broadens with increase in $\tilde{z}$ because of isotropization, while the amplitude on the axis decreases because of the nonlinear absorption. With the passage of time, the medium heats up, and the focus is shifted toward the source. The solid curve in Fig. 5 represents the dependence of the coordinate point of the "collapse" (focus) $\tilde{z}_f$ on the time $\tilde{\theta}$. We note that the velocity of the focus decreases with the passage of time.

In the case in which the scale of the absorption is large in comparison with the scale of the self-focusing, it suffices to write Eq. (16) in the following approximation

$$\frac{\partial}{\partial\theta}\left(\frac{1}{f}\frac{\partial^2 f}{\partial z^2}\right)=\frac{\mathrm{sgn}(\delta)\Pi^3}{f^5}. \qquad (16')$$

The convenience of this approximation lies in the fact that at $K=0$ (plane wavefront at $z=0$) the problem (16'), (17) has a self-similar solution $f(z,\ \theta)=\varphi(\zeta)$, where $\zeta=\Pi^{3/2}z\theta^{1/2}$. Actually, substituting $f=\varphi(\zeta)$ in the given formula, we obtain

$$\frac{1}{\varphi}\frac{d^2\varphi}{d\zeta^2}+\frac{\zeta}{2}\frac{d}{d\zeta}\left(\frac{1}{\varphi}\frac{d^2\varphi}{d\zeta^2}\right)=\frac{\mathrm{sgn}(\delta)}{\varphi^5},$$

$$\varphi|_{\zeta=0}=1,\ d\varphi/d\zeta|_{\zeta=0}=0.$$

Thus the problem is reduced to the solution of the ordinary differential equation relative to the function $\varphi(\zeta)$. Figure 6 shows the results of the calculation of φ at $\delta<0$. It is seen that the beam "collapses" at $\zeta\approx1.5$. The corresponding dependence of the coordinate of the focus on the time $\tilde{z}_f^*=1.5\tilde{\theta}^{-1/2}$ is given in Fig. 5 by the dashed curve. At $\tilde{\theta}\gtrsim3$, the value of is less than z_f^* by a factor of two, i.e., the approximate self-similar solution at $\tilde{\theta}\gtrsim3$ is entirely satisfactory. The corresponding approximation of the dependences of (13) and (14) has the form $A_0/p_0\approx f^{-1}$, $a_0/a\approx f$.

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