Theory of gradient instabilities of the gaseous galactic disk and of rotating shallow water

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A theory is derived for the stability of a rotating gaseous disk and for a layer of shallow water in the case in which there are discontinuities in the rotation velocity, the sound velocity, and the surface density at the radius $r = R$. Various instabilities of subsonic and supersonic flows are analyzed. It is shown that the system of linearized dynamic equations of the gaseous galactic disk is identical to that for rotating shallow water in devices of the Spiral' type.

1. INTRODUCTION

The present study has two purposes: to demonstrate that the spiral structure of the Galaxy is generated by gradient instabilities, and to derive a theory for the gradient instabilities of rotating shallow water for the case in which viscosity effects are negligible.

It is natural to ask why such very different entities as the gaseous disk of the Galaxy and shallow water are taken up in the same paper. The reason is that the instabilities of the gaseous galactic disk and those of rotating shallow water (the latter instabilities have been studied on the Spiral' devices in the plasma physics division at the Kurchatov Institute of Atomic Energy, Moscow) are described by the same system of differential equations. Although this is understandable—shallow water can be viewed as a two-dimensional gasdynamic system[1] analogous to the gaseous galactic disk—there are two significant differences: Bottom viscosity effects operate in an experimental device, but there are no such effects in galaxies, while galaxies have self-gravitation forces which do not operate in shallow water.

The more detailed study of these systems which we are reporting here has led us to the following conclusions: (1) The linearized dynamic equations of the gaseous galactic disk, with self-gravitation, can easily be transformed into the shallow-water equations. (2) In the Spiral' devices, the Ekman number[2] E_ν is small, and the viscosity affects the flow over a time $\tau_{sp} \gg \tau_{in}$, where τ_{in} is the rise time of the instability (Sec. 2).

To prove that the systems of equations are identical is equivalent to asserting that the mechanisms for the gradient instabilities in the gaseous disk of the Galaxy can be modeled in the Spiral' devices. The Galaxy, however, is a complex system: In addition to the gaseous disk, it contains several stellar subsystems which differ in morphological features. The gravitational potential of the steady-state system is determined in the gaseous disk by the resultant effect of all the subsystems of the Galaxy. In this spirit one could say that the rotation of the gaseous disk is externally controlled, by the stellar population, just as the rotation of a layer of shallow water in a Spiral' device is controlled by the bottom.[1)] The perturbed gravitational potential in the gaseous galactic disk, however, is determined exclusively by the perturbed gas density; the perturbations of the density of stars are negligible here (Sec. 3).

The role played by the "external" gravitational force of the stellar subsystems is thus limited to one of creating the observed rotation curve of the gaseous disk. The self-gravitation force, on the other hand, does not (as we have already mentioned) alter the structure of the equations if we introduce a simple redefinition: We replace the sound velocity c_s by the characteristic propagation velocity of the perturbations in a self-gravitating gaseous medium, c_g (Sec. 4). A similar redefinition is used in the case of shallow water:[1] The sound velocity in the water is replaced by the characteristic propagation velocity of surface-density perturbations in the water. The latter velocity is much smaller than the former. Correspondingly, the relation $c_g \ll c_s$ holds near the Sun, i.e., in the part of the gaseous disk fairly far from the center of the Galaxy, at the boundary of gravitational stability. In the central part of the Galaxy, where (we believe) there is a spiral-structure generator, the self-gravitation force is small in comparison with the pressure force, and the relation $c_g \approx c_s$ holds.

We thus reach the conclusion that these differences between the two systems do not affect the mechanisms for the occurrence of the gradient instabilities. The Spiral' devices, on which the profiles of the basic properties of the galactic disk—the rotation velocity, the sound velocity, and the surface density—are modeled may thus be regarded as analog machines which reproduce the process by which the spiral structure of the Galaxy is generated.

In Sec. 4 below we prove the equivalence of the two systems of linearized equations: that describing the dynamics of shallow water in the Spiral' devices and that describing the gaseous galactic disk. We then go on to an analytic solution of this system of initial equations in Sec. 5. We assume there that there are tangential discontinuities in the rotation velocity, the density, and the characteristic perturbation propagation velocity. By analyzing the resulting dispersion relation we find the conditions for and the boundaries of the gradient instabilities, and we discuss their physics. A "positive" density gradient of course leads to a flute instability of the disk. The physical meaning of the Landau condition[4] for stabilizing the instability at a tangential velocity discontinuity is explained. A new centrifugal instability of the disk is described. This instability does not have a stability threshold in velocity; it occurs at large Mach number, $M \gg 1$, when the gradient of the angular rotation velocity is negative. The conditions for the onset of this centrifugal instability and those for the onset of the Kelvin-Helmholtz instability are fundamentally different: The sign of the velocity gradient is irrelevant in the latter case, and there is a definite Landau

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upper limit[4] on the Mach number M. The physics of these instabilities is fundamentally different and is described in Secs. 6 and 7.

The stabilizing role of a "negative" density gradient, which has been revealed in previous studies in the examples of a planar discontinuity[5] and a cylindrical discontinuity,[6] is influential in two ways in the case of the disk: It not only significantly reduces the growth rate of the instability[7] caused by a negative velocity gradient but also makes the azimuthal phase velocity of a spiral density wave substantially lower than the value found in the calculations of Ref. 7, which ignore the density gradient. As a result, the quantitative (severalfold) difference between the theoretical[7] and experimental[8,9] values of the phase velocity of the wave disappears. We ascribe this difference to the difference between the propagation velocities of the nonlinear wave observed in the experiments of Refs. 8 and 9 and that calculated in the linear approximation in Ref. 7. As the result of the substantial "retardation" of the spiral wave due to the incorporation of the density gradient, the calculated corotation radius (i.e., the radius at which the wave velocity is equal to the rotation velocity of the disk) is transferred from the "generator" region ($r \approx 1$ kpc) to the periphery of the Galaxy. The latter situation corresponds better to the standard ideas about the location of the corotation radius in the Galaxy.[10]

In the Conclusion of this paper, we compare the theoretical results with experimental results and observational data on the spiral structure of the Galaxy.

2. THE NEGLIGIBLE EFFECT OF MOLECULAR VISCOSITY ON THE MECHANISM FOR GRADIENT INSTABILITIES IN SHALLOW WATER IN A SPIRAL' DEVICE

The estimates below apply only to the first of the Spiral' devices, in which the generation of the spiral arms of the Galaxy was modeled. This apparatus and the experimental results are described in Refs. 8 and 9.

Experiments on rotating shallow water have several numerical parameters whose values (in comparison with unity) determine the relative role played by the viscosity. First, there is the Ekman number[2]

$$E_\nu \equiv \frac{\nu}{\Omega_0 H^2}, \tag{1}$$

where ν is the kinematic molecular viscosity coefficient,[1] Ω_0 is the angular rotation velocity of the bottom of the vessel, and H is the depth of the liquid. In the Spiral' apparatus, the depth H was varied from 0.2 to 0.4 cm; with the value[1] $\nu = 0.01$ cm^2/s for water, we thus have $(E_\nu)_{\max} \approx \frac{1}{4}[\Omega_0(r)]^{-1}_{\min}$. In experiments with a "rotating periphery" the value $[\Omega_0(r)]_{\min} \approx 2$ s^{-1} prevailed, so the maximum Ekman number was $(E_\nu)_{\max} \approx 1/8$; i.e., viscosity effects were slight.

By definition,[2] the Ekman number is $E_\nu \equiv \delta^2/H^2$, where $\delta = (\nu/\Omega)^{1/2}$ is the depth of the Ekman layer. Although it is relatively small, $\delta^2 \ll H^2$, the Ekman layer could effectively alter the momentum of the entire liquid layer, of depth H, over a time scale τ_{sp} in our experiments. This time is called the "spindown time" (the "characteristic time for viscous damping") or the "spinup time" (the "characteristic time for viscous onset"). In the experiments of Refs. 8 and 9, a sharp change in the rotation velocity Ω_1 was accompanied by a change in the number of modes: An m_1-arm spiral was replaced by an m_2-arm spiral corresponding to the new rotation conditions. The time scale of the relaxation to a new steady state is τ_{sp}. From the equation of motion we find, in order of magnitude,

$$\frac{\partial v_r}{\partial t} \sim \nu \frac{\partial}{\partial z}\left(\frac{\partial v_r}{\partial z}\right). \tag{2}$$

Since v_r depends on z only in the characteristic interval[2] $(0, \delta)$, we can use the approximation

$$\partial v_r/\partial z \sim (\partial v_r/\partial z)\,\theta(\delta - z),$$

where $z > 0$, and $\theta(\delta - z)$ is the unit step function. Integrating (2) over z from 0 to H, we find

$$\frac{\partial}{\partial t} v_r H \sim \nu \left.\frac{\partial v_r}{\partial z}\right|_0^\delta \sim \nu \frac{v_r}{\delta}.$$

Multiplying the latter equality by the density ρ and by the area of the bottom, S, we find an equation for the change in the radial momentum $\mathcal{P}_r = \rho S H v_r$:

$$\frac{\partial \mathcal{P}_r}{\partial t} \sim \frac{\mathcal{P}_r}{\tau_{sp}}, \qquad \tau_{sp} \sim \frac{H\delta}{\nu} \sim \frac{H}{(\nu\Omega)^{1/2}}. \tag{3}$$

Let us estimate the time scale for the relaxation to a steady state, for the $m = 2$ mode, for example, and that for a change in the number of arms in the case in which the angular rotation velocity of the periphery changes abruptly by an amount $\Delta\Omega$. In the first case ($m = 2$), this time is clearly longer than $(\tau_{sp})_{\min} \approx 0.47$ s^{-1}: $\tau = \tau_{\min}$ at $\Omega = \Omega_1 = 18$ s^{-1} and $H \approx 0.2$ cm (in the central part of the apparatus, we have $H > 0.2$ cm). In the second case, for $\Delta\Omega \approx 2$ s^{-1}, we have $\tau_{sp} \approx 1.4$ s. The instability time τ_{in} turns out to be much shorter than these values of τ_{sp}: $\tau_{in} \ll \tau_{sp}$. The instability time depends on the extent to which the velocity change is not actually discontinuous. If this smearing is a consequence of molecular viscosity (a laminar Ekman layer), then we have $\delta_{lam} \sim (\nu/\Omega_1)^{1/2} \approx 2.4 \cdot 10^{-2}$ cm, i.e., $\delta_{lam} \ll \lambda$, where λ is the radial wavelength ($\lambda \approx 6$ cm for $m = 2$), $\gamma \approx \Omega_1$ ($\Omega_1 \approx 18$ s^{-1} for $m = 2$), and $\tau_{in} \sim 1/\gamma \approx 5 \cdot 10^{-2}$ s.

The arguments presented above to justify ignoring the minor role played by viscous friction during the onset of an instability of the shear flow in the Spiral' devices were based on experiments with dyed water with $\nu \approx 0.01$ cm^2/s. It was mentioned in Ref. 11 that a tenfold increase in the viscosity of the working solution did not qualitatively alter the spiral pattern. Only a greater increase in the viscosity would erase the spiral structure.

The question of whether turbulent viscosity affects the formation of structures in the Spiral' devices was answered in the negative by Antipov *et al.*,[12] who showed that the "viscous lifetime" of the structures is determined entirely by the laminar viscosity.

3. PROOF THAT THE PERTURBED GRAVITATIONAL POTENTIAL $\tilde{\Psi}$ DEPENDS ONLY ON THE PERTURBED SURFACE DENSITY OF THE GASEOUS COMPONENT OF THE GALACTIC DISK, $\tilde{\sigma}_g$

Our purpose in this section of the paper is to prove that the condition

$$\tilde{\sigma}_g/\tilde{\sigma}_* \gg 1, \tag{4}$$

holds, where $\tilde{\sigma}_*$ is the perturbed surface density of the stellar disk of the Galaxy. If condition (4) holds over the entire region in which the spiral structure of the Galaxy is observed, then the function

$$\tilde{\Psi}=-2\pi G(\tilde{\sigma}_g+\tilde{\sigma}_*)/|k| \qquad (5)$$

determined from the Poisson equation will actually depend on $\tilde{\sigma}_g$ alone.

We will now show that the inequality (4) follows from the hydrodynamic concept of the generation of spiral density waves[13] and from observational data. Let us assume that in the central part of the Galaxy, at $r\approx 1$ kpc, the observed sharp gradient in the rotation velocity of the gaseous disk triggers an instability of the shear flow. A large jump in the rotation velocity is clearly observed in the gaseous disk of the Galaxy[14] but not in the stellar disk. This result is not surprising: The velocity spread of the stars is too great in the stellar disk. For this reason, the velocity jump in the stellar disk must be greatly smeared out; i.e., if an instability of the shear flow does exist in the stellar gas, its growth rate γ_* is much smaller than the growth rate in the gaseous disk, γ_g: $\gamma_* \ll \gamma_g$. Because of this inequality and the exponential growth of perturbations as a result of the instability, the perturbations of the gas density, $\tilde{\sigma}_g$, are the first to grow. They cause the following perturbations in the gravitational potential:[15]

$$\tilde{\Psi}=-\frac{2\pi G\tilde{\sigma}_g R_g^{-1}}{|k|}, \qquad (6)$$

where **k** is the perturbation wave vector, and R_g is a "reduction factor" which allows for the finite thickness h_g of the gaseous disk in comparison with the wavelength of a spiral perturbation. This factor is given by

$$R_g\approx\frac{1}{1+|k|h_g/2}. \qquad (7)$$

The perturbations of the gravitational field in turn cause perturbations $\tilde{\sigma}_*$ of the stellar density. Using the results of Ref. 16, we find the following expression for $\tilde{\sigma}_*$:

$$\frac{\tilde{\sigma}_*}{\sigma_{0*}}=-\frac{k^2\tilde{\Psi}}{\varkappa^2}R_\nu(x), \qquad (8)$$

where

$$R_\nu(x)=\frac{1}{x}\left\{1-e^{-x}\left[I_0(x)+2\sum_{n=1}^{\infty}I_n(x)\left(1+\frac{1}{(\nu/n)^2-1}\right)\right]\right\}R_*. \qquad (9)$$

$I_n(x)$ is the Bessel function of imaginary argument, and the reduction factor R_* reflects the finite thickness of the stellar disk in comparison with the length of the spiral wave of the perturbation (the factor R_* was omitted from Ref. 16). Depending on the value of $|k|h_*/2$, where h_* is a scale thickness of the stellar disk, we have[17-19]

$$R_*\approx\begin{cases}\dfrac{1}{1+|k|h_*/2}, & |k|h_*/2\ll 1,\\[2ex] \dfrac{2}{|k|h_*}, & |k|h_*/2\gg 1.\end{cases} \qquad (10)$$

Since the inequality $\pi h_*/\lambda_s>1$ holds for the stellar disk, we should use the second line in (10). [In the absence of strong inequalities, i.e., for the situation which prevails in the Galaxy, it would be better to use the plots of $R_*(|k|h_*)$ which were given in Refs. 16–18 (see also Ref. 15).] The same comment applies to the estimate of $R_g(|k|h_g)$. In (8) and (9) we used the notation

$$x=k^2c_{r_*}^2/\varkappa^2,\quad \nu^2=(\omega-m\Omega_0)^2/\varkappa^2,\quad \varkappa^2=4\Omega_0^2\left(1+\frac{r}{2}\frac{\Omega_0'}{\Omega_0}\right), \qquad (11)$$

where $\Omega_0(r)$ is the angular rotation velocity of the disk, c_{r_*} is the spread in the radial velocities of the stars in the galactic disk, and $\varkappa$ is the ecliptic frequency. Since observations reveal $\tilde{\sigma}_*/\sigma_{0_*}\ll 1$ in the galactic disk, the oscillations of the density of the stellar galactic disk are described well by the linear stability theory. For this reason, the following expression was adopted for the stellar disk in the papers cited above:

$$\tilde{\sigma}_*(r,\varphi,t)\sim\exp\left[i\left(\int^r k_r(r')\,dr'+m\varphi-\omega t\right)\right],$$

where ω is a natural oscillation frequency of the stellar disk, and m is the index of the azimuthal mode (the number of spiral arms).

The rotation curve of the Galaxy[14] describes a nearly Keplerian decay ($\Omega_0\propto r^{-3/2}$) in the region $r_c\approx 1$ kpc (Ref. 20),[2)] while in the large interval of r including the vicinity of the Sun we have $\Omega_0\propto r^{-1}$. We thus have[3)] (see Ref. 21 and the bibliography there)

$$\begin{aligned} &r_c\approx 1\text{ kpc: } \varkappa_c^2=\Omega_{0c}^2,\ \Omega_{0c}=V_{0c}/r_c,\\ &V_{0c}\approx 200\text{ km/s},\ \lambda_c\approx 1{,}2\text{ kpc},\ c_{r_*c}\approx 100\text{ km/s};\\ &r_\odot\approx 10\text{ kpc: } \varkappa_\odot^2=2\Omega_{0\odot}^2,\ \Omega_{0\odot}=V_{0\odot}/r_\odot,\\ &V_{0\odot}\approx 220\text{ km/s},\ \lambda_\odot\approx 2\text{ kpc},\ c_{r_*\odot}\approx 50\text{ km/s}.\end{aligned} \qquad (12)$$

Using these data and (11), we find

$$x_\odot>x_c\approx\pi^2\gg 1.$$

Using the asymptotic expression[22] for the Bessel function $I_n(x)$ at large x,

$$I_n(x)\approx\frac{e^x}{(2\pi x)^{1/2}}\left[1+O\left(\frac{1}{x}\right)\right] \qquad (13)$$

and assuming [as follows from (12)] $\nu\sim 1$, we see that all the terms in square brackets in (9) are small in comparison with unity. Substituting (6) and (9) into (8), we finally find

$$\frac{\tilde{\sigma}_*}{\tilde{\sigma}_g}=\frac{G\sigma_{0*}\lambda}{c_{r_*}^2}\frac{R_*}{R_g}. \qquad (14)$$

This expression gives the relative size of the stellar component of the perturbed gravitational spiral potential (in comparison with that of the gaseous component). We will now show that this stellar component is negligible.

Here are values of some other parameters [other than those listed in (12)] of the gaseous and stellar disks of the Galaxy (see Ref. 21 and the bibliography there): In the central region, at $r_c\approx 1$ kpc, we have

$$\begin{aligned}&(h_g)_c\approx 0{,}06\text{ kpc},\ (\sigma_{0*})_c\approx 300\ M_\odot/\text{pc}^2,\\ &(\sigma_{0g})_c\approx 6\ M_\odot/\text{pc}^2,\ c_{sc}\approx 20\text{ km/s}.\end{aligned} \qquad (15)$$

Near the Sun, at $r_\odot \approx 10$ kpc, we have

$$(h_g)_\odot \approx 0{,}14 \text{ kpc}, \quad (\sigma_{0*})_\odot \approx 80\, M_\odot/\text{pc}^2,$$

$$(\sigma_{0g})_\odot \approx 8 M_\odot/\text{pc}^2, \quad c_{*\odot} \approx 8 \text{km/s}.$$

Using numerical values of the parameters in (12) and (15) [and numerical values of the reduction factors $R_* \approx \lambda/(\pi h_*)$ and $R_g \approx (1+\pi h_g/\lambda)^{-1}$ $(R_*)_c \approx 0.25$ and $(R_g)_c \approx 0.86$; $(R_*/R_g)_c \approx 0.29$; $(R_*)_\odot \approx 0.45$, $(R_g)_\odot \approx 0.76$, and $(R_*/R_g)_\odot \approx 0.59$], we find from (14)

$$\left(\frac{\tilde{\sigma}_*}{\tilde{\sigma}_g}\right)_c \approx 0{,}04; \quad \left(\frac{\tilde{\sigma}_*}{\tilde{\sigma}_g}\right)_\odot \approx 0{,}17. \tag{16}$$

On the basis of (16), we will be ignoring the effect of the perturbed stellar component on the evolution of spiral perturbations.

4. PROOF THAT THE SYSTEM OF LINEARIZED DYNAMIC EQUATIONS OF THE GASEOUS GALACTIC DISK IS EQUIVALENT TO THE CORRESPONDING SYSTEM OF EQUATIONS FOR ROTATING SHALLOW WATER IN THE SPIRAL' DEVICES

Having shown that $\tilde{\Psi}$ depends primarily on $\tilde{\sigma}_g$, we can write the system of linearized dynamic equations of the gaseous galactic disk as follows:

$$\frac{\partial v_r}{\partial t} + \Omega_0 \frac{\partial v_r}{\partial \varphi} - 2\Omega_0 v_\varphi = -\frac{\partial}{\partial r}(c_{g0}^2 \eta), \tag{17}$$

$$\frac{\partial v_\varphi}{\partial t} + \Omega_0 \frac{\partial v_\varphi}{\partial \varphi} + \frac{\varkappa^2}{2\Omega_0} v_r = -\frac{1}{r}\frac{\partial}{\partial \varphi}(c_{g0}^2 \eta), \tag{18}$$

$$\frac{\partial \eta}{\partial t} + \Omega_0 \frac{\partial \eta}{\partial \varphi} + \frac{\partial v_r}{\partial r} + (1 + r \ln' \sigma_0)\frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} = 0. \tag{19}$$

Here

$$c_{g0}^2 \equiv c_{*0}^2 - \frac{2\pi G \sigma_0}{|k| R_g}, \quad c_{*0}^2 \equiv \frac{dp_0}{d\sigma_0}, \quad \eta \equiv \frac{\sigma}{\sigma_0}, \tag{20}$$

where the prime means differentiation with respect to r, the subscript 0 means a steady-state value (as above), and we are omitting the tilde (˜) from the perturbed quantities. We will be using the tilde below only to specify the amplitudes of perturbed quantities. In writing (17) and (18) we used the linearized equation of state

$$p/\sigma_0 = c_{*0}^2 \eta. \tag{21}$$

If we make the substitutions

$$c_{g0}^2 = c_{w0}^2 \equiv g H_0, \quad \eta = H/H_0, \tag{22}$$

in Eqs. (17)–(19), we obtain the system of linearized dynamic equations of rotating shallow water.[23] This is the system of equations which is used in describing small perturbations on shallow water in the Spiral' devices on the f plane.[24]

5. ANALYTIC SOLUTION OF THE SYSTEM OF EQUATIONS (17)–(19) IN THE CASE OF TANGENTIAL DISCONTINUITIES IN THE ROTATION VELOCITY, THE SOUND VELOCITY, AND THE SURFACE DENSITY

Since the coefficients in the original system of equations, (17)–(19), do not depend on φ or t, we seek a solution in the form

$$A(r, \varphi, t) = \tilde{A}(r) \exp[i(m\varphi - \omega t)]. \tag{23}$$

Equations of motion (17), (18) can then be reduced to the single equation

$$\frac{d}{dr}(c_{g0}^2 \tilde{\eta}) = \frac{2m\Omega_0}{r\hat{\omega}} c_{g0}^2 \tilde{\eta} + (\hat{\omega}^2 - \varkappa^2)\tilde{\xi}, \tag{24}$$

and continuity equation (19) can be rewritten as

$$\frac{d}{dr}(r\sigma_0 \tilde{\xi}) = -r\sigma_0 \left[2\frac{m\Omega_0}{\hat{\omega}}\frac{\tilde{\xi}}{r} + \left(1 - \frac{m^2}{r^2}\frac{c_{g0}^2}{\hat{\omega}^2}\right)\tilde{\eta}\right]. \tag{25}$$

In (24) and (25) we are using the notation

$$v_r \equiv \frac{d\xi}{dt} = \left(\frac{\partial}{\partial t} + \frac{v_{0\varphi}}{r}\frac{\partial}{\partial \varphi}\right)\xi = -i\hat{\omega}\xi, \quad \hat{\omega} \equiv \omega - m\Omega_0. \tag{26}$$

According to observational data (see, for example, Ref. 14 and the references there), there is a sharp decay of the rotation curve of the gaseous component near the radius $R \approx 0.7$ kpc (more precisely, at distances $R \pm 0.4$ kpc). The distance $R \approx 0.7$ kpc from the center is a significant one because it is here that we find the edge of the central gaseous disk, whose surface density σ_{g1} is two orders of magnitude greater than the surface density σ_{g2} of the gas for $r > R$. Further from the center, the surface gas density remains essentially constant.

We will therefore assume that the angular rotation velocity $\Omega_0(r)$, the sound velocity $c_{g0}(r)$, and the surface gas density $\sigma_0(r)$ change abruptly at $r = R$:

$$\begin{aligned} &\Omega_0(r) = \Omega_1 \equiv \text{const}, \quad \sigma_0(r) = \sigma_1 \equiv \text{const}, \\ &c_{g0}(r) = c_{g1} \equiv \text{const} \quad \text{for } r < R, \\ &\Omega_0(r) = \Omega_2 \equiv \text{const}, \quad \sigma_0(r) = \sigma_2 \equiv \text{const}, \\ &c_{g0}(r) = c_{g2} \equiv \text{const} \quad \text{for } r > R. \end{aligned} \tag{27}$$

Integrating Eqs. (24) and (25) over the radial shell $(R - \varepsilon, R + \varepsilon)$, and then taking the limit $\varepsilon \to 0$, we find the following matching conditions at the discontinuity:

$$[\tilde{\eta} c_{g0}^2 + R\Omega_0^2 \tilde{\xi}]_{R-0}^{R+0} = 0, \quad [\tilde{\xi}\sigma_0]_{R-0}^{R+0} = 0. \tag{28}$$

We can now reduce the system of two first-order ordinary differential equations in (24) and (25) to a single second-order ordinary differential equation. The solution of the latter equation will contain two arbitrary constants, which we will determine from the two matching conditions (28). The equation which we are seeking has constant coefficients on the two sides of the discontinuity, for $r < R$ and $r > R$. We will make use of this circumstance in deriving this equation.

From Eq. (24) we find

$$\tilde{\xi} = \frac{c_{g0}^2}{\hat{\omega}^2 - \varkappa^2}\left(\frac{d\tilde{\eta}}{dr} - \frac{2m\Omega_0}{r\hat{\omega}}\tilde{\eta}\right). \tag{29}$$

We substitute this expression for $\tilde{\xi}$ (and the equation for $\tilde{\xi}'$, found in the obvious way) into (25). After several straightforward manipulations, we find a differential equation for Bessel functions of imaginary argument:

$$\tilde{\eta}'' + \frac{1}{r}\tilde{\eta}' - \left(k^2 + \frac{m^2}{r^2}\right)\tilde{\eta} = 0, \quad k^2 \equiv (4\Omega_0^2 - \hat{\omega}^2)/c_{g0}^2. \tag{30}$$

The general solution of Eq. (30) is[22]

$$\tilde{\eta} = Z_m(ikr) = C_1 I_m(kr) + C_2 K_m(kr).$$

Since we have $I_m(x) \to \infty$ as $x \to \infty$ and $K_m(x) \to \infty$ as $x \to 0$, we have the following solutions on the two sides of the discontinuity:

$$\eta_1=C_1I_m(k_1r),\ r<R;\ \eta_2=C_2K_m(k_2r),\ r>R, \tag{31}$$

where

$$k_{1,2}^2=[4\Omega_{1,2}^2-(\omega-m\Omega_{1,2})^2]/c_{g1,2}^2.$$

Using the two matching conditions (28) [joining the solutions (31) at the radius $r = R$], we obtain a system of two homogeneous transcendental equations, for which the unknown functions are two unknown coefficients of the solutions (31). The requirement that the solution of this system of homogeneous equations be nontrivial is equivalent to the requirement that its determinant be zero. We thus find the following dispersion relation (Appendix I):

$$k_1^2\alpha_2-k_2^2\alpha_1Q\mu^2+\frac{M^2}{R^2}\alpha_1\alpha_2(1-Qq^2)=0, \tag{32}$$

where (the prime means differentiation with respect to the argument of the Bessel function)

$$\alpha_1=\frac{2m}{x-m}-k_1R\frac{I_m'(k_1R)}{I_m(k_1R)},$$

$$\alpha_2=\frac{2mq}{x-mq}-k_2R\frac{K_m'(k_2R)}{K_m(k_2R)}, \tag{33}$$

$$M=R\Omega_1/c_{g1},\ q=\Omega_2/\Omega_1,\ Q=\sigma_1/\sigma_2,$$
$$\mu=c_{g2}/c_{g1},\ x=\omega/\Omega_1.$$

In this new notation we have

$$k_1=\frac{M}{R}[4-(x-m)^2]^{1/2},\quad k_2=\frac{M}{\mu R}[4q^2-(x-mq)^2]^{1/2}. \tag{34}$$

The parameter M represents the Mach number near the discontinuity (calculated from the "inner" velocity, i.e., that in the region $r = R - 0$), while q, Q^{-1}, and μ represent the ratios of the values of the angular velocity, the surface density of the gas, and the spread in the gas velocities in the outer region ($r>R$) to their values in the inner region ($r<R$). For $Q=\mu=1$, the dispersion relation (32) becomes the dispersion relation of Refs. 7, since the surface density σ_0 and the velocity spread c_{g0} were assumed in Ref. 7 to remain constant over the entire radius of the disk, and self-gravitation was ignored. As we will see below (see also Ref. 3), incorporating the jump in the surface density and in the spread in gas velocities along with the jump in the angular velocity of the gas leads to some qualitatively new physical effects.

6. GRADIENT INSTABILITIES AT SMALL MACH NUMBERS, $M \ll 1$

It can be seen from (33) that the relations $k_1R\sim M\ll 1$ and $k_2R\approx Mq/\mu$ hold. At sufficiently small Mach numbers, $M\ll\mu/q$ and $k_2R\ll 1$, however, by expanding the Bessel functions $I_m(k_1R)$ and $K_m(K_2R)$ for small argument,

$$\frac{I_m'(k_1R)}{I_m(k_1R)}\sim\frac{m}{k_1R};\quad \frac{K_m'(k_2R)}{K_m(k_2R)}\sim-\frac{m}{k_2R},$$

we find

$$\alpha_1=-m\left(1-\frac{2\Omega_1}{\hat\omega_1}\right),\quad \alpha_2=m\left(1+\frac{2\Omega_2}{\hat\omega_2}\right);\quad \hat\omega_{1,2}=\omega-m\Omega_{1,2}.$$

After the values of α_1 and α_2 are substituted into dispersion relation (32), the latter becomes

$$(1+Q)x^2-2[(m-1)+(m+1)qQ]x$$
$$+m[(m-1)+q^2Q(m+1)]=0. \tag{35}$$

The solution of this equation is

$$x_{1,2}=(1+Q)^{-1}\{m(1+qQ)+(qQ-1)$$
$$\pm i[m^2Q(1-q)^2-(1-qQ)^2$$
$$-m(Q-1)(1+q^2Q)]^{1/2}\}. \tag{36}$$

The reason why the parameter μ does not appear in the dispersion relation (35), which we have restricted to the zeroth order of the expansion in the parameter M, is obvious. The approximation $M\ll 1$ corresponds to $c_g\to\infty$; in this case the factor by which one "infinity" is "greater than" another is irrelevant. When subsequent terms in the expansion in M are taken into account, we find μ^2 in the equation, but that refinement goes beyond the scope of our approximation.

For $Q=1$, we naturally find from (36) the solution which was found in Ref. 7:

$$x_{1,2}=\frac{1}{2}\{m(1+q)+(q-1)\pm i[(m^2-1)(1-q^2)]^{1/2}\},\quad Q=1. \tag{37}$$

It can be seen from (37) that an instability occurs at any value $q\neq 1$. This is the Kelvin-Helmholtz instability, for whose occurrence it does not matter which region—inner or outer—is rotating more rapidly. The physics of this instability can be outlined as follows:[15] Our approximation $M\ll 1$ corresponds to the limit of an incompressible fluid, $c_g\to\infty$. We can therefore use the Bernoulli equation[1] $v^2/2+p/\sigma=$ const. The functions decay exponentially on each side of the discontinuity,[4] so a perturbation of the flow occurs in a narrow region near $r=R$, i.e., in a "cylindrical shell." We denote the regions inside and outside the circle of radius R as regions I and II. Since the flow in the "cylinder" must be conserved, the velocity is higher beyond the "hump" (in region II) than in the neighboring regions. Consequently (according to the Bernoulli equation), the pressure beyond the hump is lower, and the hump will grow.

We now consider the case of a rigid-body rotation of the entire system ($q=1$), which has an arbitrary density gradient $Q\neq 1$. From (36) we have

$$x_{1,2}=(1+Q)^{-1}\{m(1+Q)+(Q-1)$$
$$\pm i[-(1-Q)^2+m(1-Q^2)]^{1/2}\},\quad q=1. \tag{38}$$

It follows from (38) that under the condition

$$Q<1 \tag{39}$$

a flute instability occurs: An outer shell (of higher density) exerts pressure on an inner shell (of lower density).

The solution (36) with $q\neq 0$ and $Q<1$ thus describes two instabilities: the Kelvin-Helmholtz instability and the flute instability. By varying the parameters q and Q one can strengthen or weaken either instability. For example, one can suppress the flute instability by arranging a centrifugal force to oppose the gravitational force, i.e., by preparing a system with $q<1$. Correspondingly, one can stop the Kelvin-Helmholtz instability by introducing a negative density gradient, i.e., by preparing a system with $Q>1$. An example of this stabilization is given below.

Strictly speaking, it is generally not legitimate to speak in terms of separate Kelvin-Helmholtz and flute instabilities. Solution (36) describes the onset of a shear-flute instability or a gradient instability at $M \ll 1$.

a) Case with $Q \gg 1$ and $q \ll 1$. Let us see how the Kelvin-Helmholtz instability ($q \neq 1$) is suppressed when there is a negative density gradient (i.e., a drop in the density), corresponding to $Q > 1$. So that the changes will be perceptible, we assume $Q \gg 1$. For definiteness, we adopt values for the parameters q and Q which are the same as those in our Galaxy: $q \approx 0.1$ and $Q \approx 100$. (The results which follow, however, will obviously not apply to either our Galaxy, in which the relation $M \gg 1$ holds, or to other, similar spiral galaxies; the author knows of no such galaxies with $M \ll 1$.)

We thus assume

$$q \ll 1, \; Q \gg 1, \; qQ \gg 1, \; q^2 Q \sim 1. \tag{40}$$

Using (40), we find from (36)

$$x_{1,2} = q(m+1) \pm iQ^{-1/2}[m^2(1-q)^2 - (m-1)q^2 Q - m]^{1/2}, \tag{41}$$

from which we in turn find the instability condition:

$$m^2(1-q)^2 > (m+1)q^2 Q + m \tag{42}$$

or, in the case $q \ll 1$,

$$q^2 Q < \frac{m-1}{m+1} m. \tag{43}$$

It follows from (43) that an instability with $m = 0$ or 1 is not possible. An instability with $m = 2$ occurs under the condition $q^2 Q < 2/3$, etc.

A large negative density gradient thus stabilizes the system. From (41) we find the azimuthal phase velocity of a perturbation wave:

$$\Omega_p \equiv \frac{\omega}{m} = \frac{m+1}{m} \Omega_2, \quad Q \gg 1, \quad Qq \gg 1. \tag{44}$$

For comparison, the same quantity in the case $Q = 1$ is (Ref. 7)

$$\Omega_p = \frac{\Omega_1 + \Omega_2}{2} - \frac{\Omega_1 - \Omega_2}{2m}, \quad Q=1. \tag{45}$$

The difference between the phase velocities for small values of m (and for $q \ll 1$) is unimportant; for $m = 2$, for example, we have

$$\frac{(\Omega_p)_{Q \gg 1}}{(\Omega_p)_{Q=1}} \equiv \alpha_p \approx 6q,$$

and for $q \approx 0.1$ we find $\alpha_p \approx 0.6$.

However, we know from theoretical[7] and experimental[8,9] results that under the condition $M \ll 1$ modes with large $m \gg 1$ are excited. For $m \gg 1$, the difference between the phase velocities is more important: for $m \gg 1$ we have

$$\frac{(\Omega_p)_{Q \gg 1}}{(\Omega_p)_{Q=1}} \equiv \alpha_p \approx 2q.$$

For $q \approx 0.1$, the difference in phase velocities reaches 5; i.e., we find $\alpha_p \approx 0.2$. A density wave with $Q \gg 1$ (and $q \ll 1$) thus rotates at an azimuthal velocity lower than the same wave with $Q = 1$ (and $q \ll 1$) would. Correspondingly, the corotation radius is more remote in the case $Q \gg 1$ than in the case $Q = 1$.

7. GRADIENT INSTABILITIES AT LARGE MACH NUMBERS, $M \gg 1$

Writing x in the form $x_1 + iMx_2$, we find from (34)

$$k_1 \approx \frac{M^2 x_2}{R} - i\frac{M}{R}(x_1 - m), \tag{46}$$

$$k_2 \approx \frac{M^2 x_2}{\mu R} - i\frac{M}{\mu R}(x_1 - mq). \tag{47}$$

Substituting k_1 and k_2 into Eq. (32), we find

$$x_1 = \frac{m(1+Q\mu q)}{1+Q\mu}, \quad x_2 = \frac{1-Qq^2}{1+Q\mu}, \tag{48}$$

or

$$x = (1+Q\mu)^{-1}[m(1+Q\mu q) + iM(1-Qq^2)]. \tag{49}$$

In the particular case $Q = \mu = 1$, expression (49) becomes the solution of Ref. 7:

$$x = \frac{1}{2}[m(1+q) + iM(1-q^2)], \quad Q=\mu=1. \tag{50}$$

Solution (50) is fundamentally different from (37) in that while the latter describes an instability for arbitrary $q \neq 1$ the former determines an instability only for $q < 1$. Consequently, for $M \gg 1$ an instability develops only if the inner part of the system ($r < R$) is rotating at an angular velocity higher than that of the outer part. We have labeled such an instability a "centrifugal instability."[8,9] The physics of this instability is analogous to that for the flute instability, while it is fundamentally different from that of the Kelvin-Helmholtz instability. It was proved in Ref. 4 that the Kelvin-Helmholtz instability is stabilized for two-dimensional perturbations at $M > 2\sqrt{2}$; the stabilization mechanism is simple. In the case of a supersonic flow ($M \gg 1$) in a narrow shell around the circle $r = R$ (Sec. 6), a hump in region II is perceived by the flow as a constriction in a supersonic nozzle, at which the velocity does not increase (as it would in a subsonic nozzle with $M \ll 1$) but instead decreases. As a result, the pressure beyond the hump increases, pushing it into region I. This question is discussed in more detail in Appendix II.

In the particular case of a rigid-body rotation of the entire system ($q = 1$), we find from (49)

$$x = (1+Q\mu)^{-1}[m(1+Q\mu) + iM(1-Q)], \quad q=1. \tag{51}$$

This solution describes the flute instability of a supersonic flow. As we will see, the condition for the flute instability remains the same, (39); i.e., it does not depend on the value of M. In the case $M \gg 1$, as in the case $M \ll 1$, the solution (49) does not describe any specific instability which occurs for $q < 1$ or $Q < 1$. It instead describes a centrifugal-flute instability or (again) a gradient instability for $M \gg 1$. The condition for the occurrence of this instability is

$$Qq^2 < 1. \tag{52}$$

We can now determine the azimuthal phase velocity of the perturbations in the general case described by solution (49):

$$\Omega_p \equiv \frac{\mathrm{Re}\,\omega}{m} = \frac{1+Q\mu q}{1+Q\mu} \Omega_1. \tag{53}$$

In the particular case of Ref. 7 we have

$$Q=\mu=1, \quad \Omega_p=\frac{\operatorname{Re}\omega}{m}=\frac{1+q}{2}\Omega_1. \tag{54}$$

In the Galaxy we have[4)] $q\approx 0.1$, $Q\approx 100$, and $\mu\sim 0.1$; i.e., we find from (53)

$$(\Omega_p)_{Gal}\approx 0{,}36\ \Omega_1/2. \tag{55}$$

This result means that the corotation radius lies substantially further from the center than is predicted by a theory with $Q=1$ (Ref. 7):

$$(\Omega_p)_{Gal}\approx\Omega_1/2, \quad Q=1. \tag{56}$$

According to (55) and (56), the velocities Ω_p differ by a factor of about 3. The value measured for Ω_p in the experiments of Refs. 8 and 9 was several times lower than the value of Ω_p found from expression (54) (Ref. 7). Estimates show that the values of Ω_p in (53) are close to those measured in Refs. 8 and 9. In addition, the value found for Ω_p for the Galaxy from (55) corresponds better than the value from (56) to the present understanding[10] of the value of Ω_p.

Let us find the shape of the spiral pattern generated by the gradient instability beyond the radius of the discontinuity (at $r>R$). We find the following expression for the magnitude of the perturbed density σ for $r>R$ from (31) and (47):

$$\sigma\propto K_m(k_2r)e^{im\varphi}\propto r^{1/2}\exp\left\{-\frac{\Omega_1^2Rr}{c_{g1}c_{g2}}\frac{(1-Qq^2)}{(1+Q\mu)}+im\left[\varphi+\frac{\Omega_1 r}{c_{g2}}\frac{(1-q)}{(1+Q\mu)}\right]\right\}. \tag{57}$$

From solution (57) we can draw two conclusions.

1. The necessary condition for a finite solution is the same as the condition for the occurrence of the gradient instability at $M\gg 1$, (52).

2. The density waves are lagging spirals only in a system in which the angular rotation velocity falls off with increasing radius, i.e., for $q<1$. The latter is a necessary condition for the occurrence of the centrifugal instability in the system.

From (57) we easily find the radial wavelength:

$$\lambda_r=\frac{2\pi}{k_r}=2\pi\frac{c_{g2}}{\Omega_1 m}\frac{(1+Q\mu)}{(1-q)}. \tag{58}$$

8. CONCLUSION

1. It has been shown that the perturbed gravitational potential of the Galaxy in the $z=0$ plane is determined primarily by the perturbed surface density of the gaseous disk. The perturbed surface density of the stellar disk is negligible here.

2. Through the introduction of a gravitating sound velocity c_g, the linearized dynamic equations for a gravitating gaseous disk have been reduced to the corresponding equations for rotating shallow water.

3. The bottom viscosity has only a minor effect on the mechanism for the generation of spiral density waves in the shallow water in the Spiral' devices. These devices may be thought of as analog machines for modeling the generation of the spiral arms in the gaseous disk of the Galaxy.

4. The stability of a rotating, gravitating, two-dimensional (planar) gaseous disk (or of rotating shallow water) with discontinuities in the sound velocity, the rotation velocity, and the surface density at $r=R$ has been analyzed. A "gradient instability" has been found. For $M\ll 1$ this instability is called a "shear-flute instability," while at $M\gg 1$ it is a "centrifugal instability."

5. In the particular case of a medium with a homogeneous density, $Q=1$, the shear-flute instability converts into a Kelvin-Helmholtz instability,[7] and the centrifugal-flute instability into a centrifugal instability.[7–9]

6. In the particular case of a uniform angular rotation velocity, $q=1$, the shear-flute and centrifugal-flute instabilities convert into the flute instability. The condition for the occurrence of the latter, (39) is the same for all M.

7. The azimuthal phase velocity of the perturbations, Ω_p, is determined by the values of q, Q, and μ. Under the experimental conditions of Refs. 8 and 9 and under the conditions in the galactic disk, the values of Ω_p are close to the theoretical value of Ω_p found from (53).

8. The perturbations take the form of lagging spirals (in the case $M\gg 1$, the angular velocity of the disk must fall off with the radius).

I wish to thank M. V. Nezlin for useful discussions.

APPENDIX I

Substituting $\tilde{\eta}$ from (31) into (29), and using the notation of (33), we find the following expressions for $\tilde{\xi}_1$ and $\tilde{\xi}_2$ for the regions $r<R$ and $r>R$, respectively:

$$\tilde{\xi}_1=C_1\frac{\alpha_1 I_m}{k_1^2R}, \quad r<R; \qquad \tilde{\xi}_2=C_2\frac{\alpha_2K_m}{k_2^2R}, \quad r>R. \tag{I1}$$

Using the second matching condition in (28) we find

$$C_1\frac{\sigma_1\alpha_1}{k_1^2}I_m=C_2\frac{\alpha_2\sigma_2}{k_2^2}K_m. \tag{I2}$$

From the first matching condition in (28) we find

$$\tilde{\eta}_1+\frac{M^2}{R}\tilde{\xi}_1=\mu^2\tilde{\eta}_2+\frac{M^2q^2}{R}\tilde{\xi}_2, \tag{I3}$$

or, with the help of (31) and (I1)

$$\left(1+M^2\frac{\alpha_1}{k_1^2R^2}\right)I_mC_1=\left(\mu^2+M^2q^2\frac{\alpha_2}{k_2^2R^2}\right)K_mC_2. \tag{I4}$$

The system of homogeneous transcendental equations in (I2) and (I4) has a nontrivial solution with respect to the unknown functions C_1 and C_2 if the determinant of this system of equations vanishes. The latter condition is the dispersion relation which we need, (32).

APPENDIX II

For $M>1$, the mechanism which was described above qualitatively, and which was originally proposed by Landau,[4)] for stabilizing the Kelvin-Helmholtz instability contradicts a basic conclusion of a paper by Syrovatskiĭ[25]—that there is no condition for stabilizing a tangential-discontinuity instability. This criticism of Landau's paper[4] was, as we know, the reason why that paper was omitted from the two-volume collection of Landau's works and why the contents of that paper were removed from the 1954 edition of Landau and Lifshitz's *Mekhaniki sploshnykh sred* (*Fluid Mechanics*) (the results in question had been included in the 1953 edition).

We can show that although Syrovatskiĭ's derivation[25] is correct for a medium of infinite size, a stabilizing effect does operate in a real, spatially bounded, supersonic flow with a tangential velocity discontinuity. This effect can be described quantitatively by a modified Landau condition.

The instability of a tangential discontinuity of a quasi-two-dimensional flow (e.g., the gaseous disks of the galaxies and shallow water) is stabilized, as was shown above (and as was shown previously in Refs. 26 and 27), in complete accordance with Landau's condition.[4]

Let us assume that we are given a flow along the x axis with a tangential velocity discontinuity: $v_0 = v_x\theta(-z)$, where θ is the unit step function. Choosing the perturbations of the density ρ and the velocity v in the form

$$\rho(x,z,t)\sim v(x,z,t)\sim\exp(ikx-\lambda|z|+\gamma t), \qquad \text{(II1)}$$

Landau showed[4] that an instability does not occur under the condition $v_0 > v_{cr}$. If the unperturbed density ρ_0 and the unperturbed sound velocity c_0 are assumed to remain constant on each side of the discontinuity, $\rho_{01} = \rho_{02} = \rho_0$, $c_{01} = c_{02} = c_0$, then we have the following result in this very simple case:

$$v_{cr}=2^{3/2}c_0. \qquad \text{(II2)}$$

It can be seen from (II1) that the wave vector $\mathbf{k}$ was chosen along the x axis in Ref. 4: $k = k_x$. Ten years later, Syrovatskiĭ[25] solved a related problem with respect to a general class of perturbations, $\mathbf{k} = \{k_x, k_y\} = \{k\cos\theta, k\sin\theta\}$, and found that an instability occurred at any value of v_0.

For $\rho_{01} = \rho_{02}$, $c_{01} = c_{02}$, the problem of the stability of a tangential discontinuity in the flow of a compressible fluid with respect to arbitrary perturbations can be reduced to the following dispersion relation [we are assuming a time dependence $\propto \exp(-i\omega t)$]:

$$k^2c_0^2\left[\frac{1}{(\omega-kv_0)^4}-\frac{1}{\omega^4}\right]=\frac{1}{(\omega-kv_0)^2}-\frac{1}{\omega^2}. \qquad \text{(II3)}$$

Taking out a common factor, which has only the real root $\omega = -kv/2$, we find the equation

$$f(x)=1, \qquad \text{(II4)}$$

where

$$f(x)\equiv\frac{1}{(x-M\cos\theta)^2}+\frac{1}{x^2},\quad x\equiv\frac{\omega}{kc_0},\quad M\equiv\frac{v_0}{c_0}.$$

This result differs from Landau's equation[4] by a factor of $\cos\theta$ (Ref. 25). Equation (II4) has four roots. All are real if the function $f(x)$ is of the form shown by the solid line in Fig. 1. If $f(x)$ is instead of the form shown by the dashed line, then Eq. (II4) has only two real roots. The two others are thus complex conjugates, and one of them describes an instability. The majorant curve is shown by the dot-dashed line. Again in this case we have all real roots, two of which are multiple:

$$x_1',\quad x_4',\quad x_2'=x_3'\equiv\frac{1}{2}M\cos\theta.$$

The critical Mach number M_{cr} is found from the equation

$$f\left(\frac{1}{2}M\cos\theta\right)=1,$$

which gives us the point at which the majorant curve is tangent to the line $f(x) = 1$. The result is

$$M_{cr}=\frac{2\sqrt{2}}{\cos\theta}. \qquad \text{(II5)}$$

Since we have $\cos\theta = k_x/|k_\perp|$, where $|k_\perp| \equiv (k_x^2 + k_y^2)^{1/2}$, we find

$$M_{cr}^2=8(1+k_y^2/k_x^2). \qquad \text{(II6)}$$

In quasi-two-dimensional systems (e.g., gaseous galactic disks and shallow water), only the "longitudinal" waves, with $k_y/k_x \ll 1$, are possible. In this case the value of M_{cr} in (II6) becomes the value found for M_{cr} by Landau.[4] Syrovatskiĭ's main comment can be summarized as follows: Arbitrary perturbations permit an analysis of the opposite limit, of "transverse" waves, with $k_y/k_x \gg 1$. It is obvious that, for

$$k_y/k_x\to\infty, \qquad \text{(II7)}$$

for example, stabilization is impossible in principle, since we find[5)] $M_{cr} \to \infty$ from (II6). In the idealized formulation of the problem—tangential velocity discontinuity in an infinite

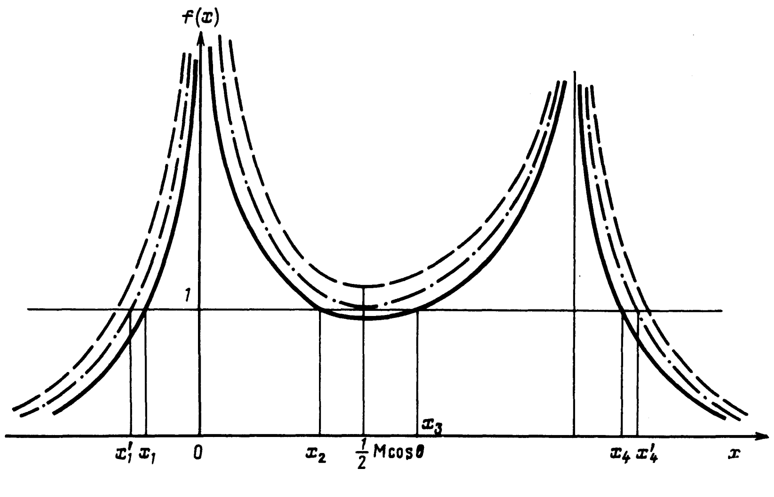

FIG. 1.

three-dimensional space—condition (II7) can be satisfied. A real situation, however, introduces two important corrections: (1) The system has finite spatial dimensions in all three directions. (2) The tangential velocity discontinuity is smeared over some distance a.

A consequence of these conditions is the existence of a limit

$$(k_y/k_x)_{max} \equiv (k_y)_{max}/(k_x)_{min}.$$

Indeed, we have $(k_x)_{min} \sim 1/L$, where L is the size of the system along the x direction, and $(k_y)_{max} \sim 1/a$, as follows from the necessary condition for an instability of a flow with a nonuniform velocity profile, $k_y a < 1$ (Ref. 28).

The instability of a "tangential discontinuity" of the velocity is thus eliminated under real conditions if the following criterion holds:

$$M > M_{cr} = 2\left[2\left(1+\frac{L^2}{a^2}\right)\right]^{1/2}. \quad \text{(II8)}$$

In practice, the condition $L^2/a^2 \gg 1$ usually holds, and in this case M_{cr} is, according to (II8), greater than $M_{cr,L}$ (Landau's) by a factor of L/a:

$$M_{cr} \approx \frac{L}{a} M_{crL}. \quad \text{(II9)}$$

We can now write the condition under which these perturbations drift:

$$1/\gamma_{max} \gg L/v, \quad \text{(II10)}$$

where $\gamma \equiv \text{Im}\,\omega$ is the growth rate of the instability of a tangential velocity discontinuity. The meaning of condition (II10) is that the perturbations in any region of the gas do not manage to grow over the time which it takes this region of the gas to traverse the system, of length L, at a velocity v: Under condition (II10), it can be assumed that this instability does not occur. When we substitute a result from Ref. 5,

$$\gamma_{max} \approx 0{,}5\,(k_x)_{max} c \approx 0{,}5 c/a,$$

into (II10), we f nd

$$M \gg 0{,}18 M_{cr}. \quad \text{(II11)}$$

The satisfaction of condition (II8) thus essentially also means the satisfaction of condition (II11).

A flow with a velocity discontinuity and a Mach number $M > M_{cr}$ is stable if the size of the flow, L, satisfies

$$L < \frac{aM}{2\sqrt{2}}. \quad \text{(II12)}$$

1) The part of the rotation curve from $\lesssim 400$ ps to $\gtrsim 1$ kpc may be an exception. On this part of the curve, a massive molecular disk ≈ 700 ps in radius, with a sharp edge, near the center of the Galaxy apparently contributes significantly to the potential. The edge of the disk is correlated with the region of a negative gradient on the curve of the rotation velocity of the Galaxy, i.e., with the region of the generator of the instability responsible for the origin of the spiral structure.[3]

2) More precisely, $r \in (R - \Lambda, R + \Lambda)$, where $R \approx 0.7$ kpc and $\Lambda \approx 0.3$ kpc (Ref. 14).

3) Presently the spiral arms can be seen from a distance $r \gtrsim 2$–3 kpc, so a value $\lambda_c \approx 1.2$ kpc at $r \approx 1$ kpc should be regarded as somewhat arbitrary at this point.

4) It follows from (20) that Q and μ are independent. In our galaxy, at a distance $r = R$ (i.e., at $r \approx 0.7$ kpc) we have $c_{g0}^2 \approx c_{s0}^2$, and μ is determined entirely by the equation of state. For a polytrope model $(p/\sigma^{\gamma_{pl}}) = \text{const}$, $\mu = Q^{(1-\gamma_{pl})/2}$ and with $\gamma_{pl} \approx 2$ we have $\mu \approx Q^{-1/2}$. This result tells us that we have $\mu = 0.1$ (with $Q = 100$). In the molecular disk, this value corresponds to a turbulence velocity $c_{s,turb} \approx 80$ km/s. If we instead substitute a non-self-consistent value $[(c_{s,turb})_{min} \approx 20$ km/s$]$ into (53), we find $\Omega_p \approx 0.26\Omega_1/2$ instead of the value of 0.36 $\Omega_1/2$ in (55). The difference is seen to be insignificant.

5) Note, however, that in the limit $k_y/k_x \to \infty$ the instability growth rate vanishes[5] $(\gamma \to 0)$. It is shown below that incorporating the growth of perturbations in drifting flows causes essentially no change in the stabilization condition found from (II6).

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