Evolution equation for the spin of a relativistic electron in the Heisenberg representation

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In this paper the evolution of the spin of an electron in the Heisenberg representation is considered based on the Dirac theory. A quantum generalization of the Bargmann–Michel–Telegdi equation is obtained.

As is well known, Bargmann, Michel and Telegdi[1] (BMT) by developing a quasi-classical theory of the electron spin, obtained an evolution equation for the spin of a relativistic particle. By virtue of its simplicity and clarity, this equation at present is widely employed to analyze the spin dynamics of electrons moving in an external electromagnetic field (cf., e.g., Ternov,[2]). The detailed derivation of this equation by means of a relativistic generalization of the well-known equation for the precession of the spin

$$\frac{d}{dt}\boldsymbol{\zeta}=\frac{e_0}{mc}[\mathbf{H}\boldsymbol{\zeta}],\ \boldsymbol{\zeta}=\langle\sigma^P\rangle \tag{1}$$

(here the electron is assumed to move in a time-independent magnetic field and $\mathbf{J}^P$ are the Pauli matrices) is presented most clearly by Berestetskii *et al.*[3] This equation for the spin dynamics is of the form

$$\frac{d}{dt}\boldsymbol{\zeta}=[\boldsymbol{\Omega}\boldsymbol{\zeta}],$$
$$\boldsymbol{\Omega}=\frac{e_0}{mc}\left\{\frac{g}{2}\frac{\mathbf{H}}{\gamma}-\frac{g-2}{2}\frac{\gamma}{1+\gamma}[\boldsymbol{\beta}[\boldsymbol{\beta}\mathbf{H}]]\right\}. \tag{2}$$

In this equation the spin vector ζ is understood to be the average over a wave packet of the unit spin vector, $\hat{\mathbf{O}}$:$\zeta = \langle\hat{\mathbf{O}}\rangle$, which characterizes the "true" spin in the Darwin sense in the particle rest-frame (cf. Ref. 4):

$$\hat{\mathbf{O}}=\rho_3\boldsymbol{\sigma}+\rho_1 c\hat{\mathbf{P}}/E-\rho_3 c^2\hat{\mathbf{P}}(\boldsymbol{\sigma}\hat{\mathbf{P}})/E(E+mc^2). \tag{3}$$

Here $\gamma = E/mc^2$, $\beta = c\langle\hat{\mathbf{P}}\rangle/E$, $\hat{\mathbf{P}} = \hat{\mathbf{p}} - e\mathbf{A}/c$, and g is the electron gyromagnetic factor. The Dirac matrices α, σ, ρ_3 and ρ_1 are related to the standard matrices γ^μ as follows:

$$\boldsymbol{\alpha}=\gamma^0\boldsymbol{\gamma},\quad \boldsymbol{\sigma}=\gamma^0\gamma^5\boldsymbol{\gamma},\quad \rho_3=\gamma^0,\quad \rho_1=-\gamma^5.$$

As was noted in Ref. 3, a similar equation in a different form was first obtained by Frenkel in 1926 (Ref. 5), proceeding from the relativistic generalization of the model of spinning top. Ternov and Bordovitsyn[6] showed that the equation of the spin dynamics (1), (2) and the Frenkel equation coincide to within inessential transformations.

Consider the quantum equation for the time evolution of the spin operator (3) in the Heisenberg representation:

$$\frac{d\hat{\mathbf{O}}}{dt}=\frac{i}{\hbar}(\hat{\mathcal{H}}\hat{\mathbf{O}}-\hat{\mathbf{O}}\hat{\mathcal{H}}), \tag{4}$$

where

$$\hat{\mathcal{H}}=c(\boldsymbol{\alpha}\hat{\mathbf{P}})+\rho_3 mc^2+\frac{\alpha}{2\pi}\mu_0\rho_3(\boldsymbol{\sigma}\mathbf{H}),$$

and the factor $\alpha/2\pi = (g-2)/2$ accounts for the anomalous magnetic moment of the electron. We obtain the expression

$$\frac{d\hat{\mathbf{O}}}{dt}=\frac{ec}{E}\left\{1-\frac{\rho_3 c(\boldsymbol{\alpha}\hat{\mathbf{P}})}{E+mc^2}\right\}[\boldsymbol{\sigma}\mathbf{H}]$$
$$+\frac{\alpha}{2\pi}\frac{e}{mc}\left([\boldsymbol{\sigma}\mathbf{H}]+\frac{c\rho_2}{E}(\boldsymbol{\sigma}\mathbf{H})\hat{\mathbf{P}}\frac{E-mc^2}{E+mc^2}\right.$$
$$\left.+\frac{c^2\hat{\mathbf{P}}(\boldsymbol{\sigma}[\mathbf{H}\hat{\mathbf{P}}])}{E(E+mc^2)}\right). \tag{5}$$

The physical interpretation of this equation is quite difficult since as is well known (cf., e.g., Ternov *et al.*[7]) in the Dirac theory the relation between the operators and classical quantities becomes complicated due to the peculiar nature of the particle motion—a rapidly oscillating vibration which was called "Zitterbewegung" by Schrödinger.[8] This motion is a result of the phenomenon of interference of various charge-conjugate states of an electron.

We shall, therefore, proceed to operators with a given parity, assuming that the even part of any operator $\hat{F}$ is determined by the expression

$$\hat{F}^{\text{even}}\equiv\{\hat{F}\}=\frac{1}{2E}(\hat{F}\hat{\mathcal{H}}+\hat{\mathcal{H}}\hat{F}). \tag{6}$$

Here we confine ourselves to consideration of an electron moving in a purely magnetic field. In the general case the replacement

$$E=|\hat{\mathcal{H}}|\to|\hat{\mathcal{H}}-e\varphi| \tag{7}$$

should be carried out.

In free motion an even operator $\{\hat{F}\}$ commutes with the Hamiltonian and does not mix states belonging to energies of different sign. This provides for the possibility of a transparent physical interpretation of operators of a definite parity since physically observable quantities can be associated with even operators which commute with the Hamiltonian.

Isolating the even part of the operator (5) we obtain

$$\left\{\frac{d\hat{\mathbf{O}}}{dt}\right\}=\frac{ec}{E}\left(1+\frac{\alpha}{2\pi}\gamma\right)[\mathbf{O}\mathbf{H}]-\frac{\alpha}{2\pi}\frac{e}{mc}\frac{c^2[\hat{\mathbf{O}}\hat{\mathbf{P}}](\hat{\mathbf{P}}\mathbf{H})}{E(E+mc^2)}. \tag{8}$$

Thus this exact Heisenberg equation for the even part of the derivative operator $d\hat{\mathbf{O}}/dt$ coincides functionally with the classical BMT equation [cf. Eq. (2) in Ref. 9]. If we further proceed to the expressions averaged over the wave packet, setting $\langle\{\mathbf{O}\}\rangle = \zeta, \langle c\mathbf{P}\rangle = \beta E$, we then obtain the classical BMT equation in the form

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$$\left\langle\left\{\frac{d}{dt}\hat{O}\right\}\right\rangle=\dot{\zeta}=\frac{ec}{E}\left(1+\frac{\alpha}{2\pi}\gamma\right)[\zeta\mathbf{H}]$$
$$-\frac{\alpha}{2\pi}\frac{e}{mc}\frac{\gamma}{1+\gamma}[\zeta\beta](\beta\mathbf{H}). \qquad (9)$$

We now consider a quantum generalization of this equation. For this purpose we observe that the variation of an arbitrary operator $\hat{F}$ in time in the Heisenberg representation can be represented in the form

$$\frac{d^2\hat{F}}{dt^2}+\frac{2i}{\hbar}\frac{d\hat{F}}{dt}\hat{\mathscr{H}}=\frac{2i}{\hbar}\left\{\frac{d\hat{F}}{dt}\right\}E, \qquad (10)$$

where the braces denote the even part of the derivative of the operator $\hat{F}$ [cf. (6)]. In accordance with this general formula, we obtain for the operator $\hat{O}$ the expression

$$\left(\frac{d\hat{O}}{dt}-\left\{\frac{d\hat{O}}{dt}\right\}\right)E=\frac{i\hbar}{2}\frac{d^2\hat{O}}{dt^2}, \qquad (11)$$

in which the right-hand side of the relation which is proportional to Planck's constant $\hbar$ describes the "Schrödinger vibration." This can be verified because the charge-conjugate states can appear only in the expression for the second derivative of $\hat{O}$ since in the limit $\hbar\to 0$ we get $d\hat{F}/dt=\{d\hat{F}/at\}$, i.e., the vibrations are absent.

The spin evolution equation (11) is an exact quantum equation.

Let us consider the range of applicability of the classical approximation of Eq. (11). Evidently, the r.h.s. of this equation can be neglected provided

$$\left|\frac{\hbar}{2E}\frac{\langle\ddot{O}\rangle}{\langle\dot{O}\rangle}\right|\ll 1 \quad \text{or} \quad \left|\frac{\hbar}{2E}\langle\dot{O}\rangle\right|\ll 1. \qquad (12)$$

Assuming furthermore, for simplicity, that the motion of the electron occurs in the plane of rotation ($\mathbf{P}\perp\mathbf{H}$), we find using (9) that

$$\left|\frac{\hbar}{2E}\frac{ecH}{E}\left(1+\frac{\alpha}{2\pi}\gamma\right)\right|=\left|\frac{\omega^{\text{prec}}}{\omega_{\text{vib}}}\right|\ll 1. \qquad (13)$$

Thus, the Heisenberg equation (11) becomes the classical BMT equation in the case when the precession frequency of the electron spin in the magnetic field is much smaller than the frequency of the "vibration." This criterion is especially transparent in the nonrelativistic case of the evolution of the spin in a strong magnetic field $H\to H_0=m^2c^3/e\hbar$. Indeed, under this assumption Eq. (13) becomes

$$\left|\frac{eH}{mc}\frac{\hbar}{2mc^2}\right|\ll 1 \quad \text{or} \quad H\ll 2H_0. \qquad (14)$$

Consequently, in a strong magnetic field the "Schrödinger vibration" becomes a substantial effect and a transition to the classical BMT equation is impossible.

Note that in a strong magnetic field when $H\to H_0$ the problem under consideration steps into ultra-quantum region since the Planck constant appearing in the expression for the energy of interaction of a magnetic dipole with the magnetic field

$$(\mu\mathbf{H})=\frac{e\hbar}{2mc}(\sigma\mathbf{H}),$$

drops out:

$$\hat{\mathscr{H}}=\hat{P}^2/2m+\mu_0(\sigma\mathbf{H}), \quad |\mu_0 H|=\frac{e\hbar}{2mc}\frac{m^2c^3}{e\hbar}\to mc^2/2. \qquad (15)$$

In this ultraquantum domain spin effects can be observed even in the zeroth order of $\hbar$.

When the criterion (15) is fulfilled the r.h.s in Eq. (11) can be set to zero and assuming that the value of the spin averages over the wave packet equals $\langle\mathbf{O}\rangle=\zeta$, one obtains the BMT equation [cf. Eq. (9) in Ref. 10].

Next, we shall display the explicit form of Eq. (11) omitting for simplicity of the subsequent estimates the terms which are due to the effect of the anomalous magnetic moment of the electron. We then obtain

$$\frac{d^2}{dt^2}\hat{O}+\frac{2i}{\hbar}\frac{d\hat{O}}{dt}\hat{\mathscr{H}}=\frac{2iec}{\hbar}\left([\hat{O}\mathbf{H}]+\frac{\hbar c}{2E}\operatorname{grad}\hat{G}\right), \qquad (16)$$

where

$$\hat{G}=[1+c\rho_3(\alpha\hat{\mathbf{P}})/(E+mc^2)](\alpha\mathbf{H}).$$

It follows from this that the quantum terms in the exact spin evolution equation are not only related to the "Schrödinger vibration," but also characterize the effect of nonuniformity of the magnetic field. Furthermore, it is evident that a transition to the classical BMT equation is feasible provided

$$\left|\frac{\hbar c\beta}{E}\frac{\operatorname{grad}H}{H}\right|\ll 1. \qquad (17)$$

This restriction requires that the magnetic field vary smoothly over the distances of the order of the Compton wavelength. In the non-relativistic approximation Eq. (16) becomes

$$G=\frac{\hbar}{mc}\operatorname{grad}(\beta\mathbf{H}). \qquad (18)$$

This accounts for the effect of terms of order $\hbar^2$ in the initial Hamiltonian. This is equivalent in the asymptotic expansion in the quasi relativistic Dirac theory in powers of the operator $\hat{\mathbf{P}}/mc$, i.e., $(\hbar/mc)$ grad (cf. Messiah[11]).

In conclusion, we emphasize once more that the BMT equation follows from the exact Heisenberg equation[10] and describes the evolution of the spin under the condition that the "Schrödinger vibration" and the gradients of the magnetic field can be neglected in accordance with the criteria (14) and (17). The corresponding terms in the Heisenberg equation are the first order terms in Planck's constant $\hbar$.

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