Statistical perturbation theory for an electromagnetic field in a medium with a rough boundary

N. P. Zhuk, O. A. Tret'yakov, and A. G. Yarovoĭ

A. M. Gor'kiĭ Kharkov State University

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A boundary-value problem of macroscopic electrodynamics, for a monochromatic field in an inhomogeneous and anisotropic medium with a rough boundary, is analyzed. This problem is basic to several applications in laser optics, solid state physics, and surface-polariton spectroscopy. A rigorous method for going over from this problem to an equivalent problem in a region with a plane interface is developed. This transformation is made with the help of "coupling" equations proposed here, which couple the values of field vectors at spatially separate points. For the case of random roughness, the presence of the roughness with respect to the average field can be described rigorously by effective-roughness operators which are assigned to a plane boundary and which incorporate the multiple scattering of waves by the roughness. A complete asymptotic expansion of these operators in powers of a small parameter characterizing the roughness height is found. The model of a uniaxial lamellar medium is studied as an example. An operator representing the equivalent impedance of the rough boundary is found for this model. The problem of the distortion of natural *s*- and *p*-polarized waves by roughness is solved.

The scattering of electromagnetic waves by a statistically rough surface originally arose in connection with radar and the theory of the propagation of radio waves along the earth's surface.[1,2] This problem has now emerged as one of major importance for laser optics and solid state physics in connection with effects accompanying the reflection of light from the surface of a nonlinear medium,[3] surface-enhanced Raman scattering,[4] and the perturbation of a surface-wave spectrum by an interface roughness.[5]

A theory for wave scattering by a nonplanar surface of fixed shape has presently been worked out in two perturbation-theory versions: a perturbation theory based on a low height of the roughness irregularities[2,6–10] and one based on their small slopes.[11] The methods proposed in the papers just cited can be used to calculate field vectors or reflection-transmission and scattering operators which determine these field vectors in any order of perturbation theory. Since these methods are limited by the requirement that the distortions of the field must be small, they are incapable of dealing with "accumulating" effects which arise in the multiple scattering of waves by irregularities and which lead to pronounced perturbations of the field.

The theory of multiple scattering by a statistically rough surface,[2,6] which arose in connection with pioneering studies by Feĭnberg[1] and Bass,[12] does not share this drawback and thus has an indisputable advantage. Various models of a medium with a rough boundary were examined with the help of this theory in Refs. 1 and 13–16. The ideas of this theory have by now been reduced to various recipes[2,16] for determining how various types of infinite sequences of scatterings by a roughness shape the electromagnetic field. Since it is not possible at the outset to evaluate the contribution of each of the various possible sequences, these recipes are empirical. Their ranges of applicability and ways to refine them are not clear. This is a weak point of the existing versions of this theory in comparison with methods which deal with scattering a finite number of times.[2,6–11] As has already been mentioned, a large number of scattering problems involve a small parameter: the height or slope of the roughness irregularities. There is the attractive possibility of formulating a multiple-scattering theory as a perturbation theory in terms of one of these small parameters.

Our goal here is to derive a systematic perturbation theory which incorporates a multiple scattering of waves by a statistically rough surface and which uses the roughness height as a small parameter. We will also apply this theory to the key problem[5] of the distortion of the spectrum of surface electromagnetic waves by an interface roughness.

Let us outline the logic of our approach. The original problem of macroscopic electrodynamics in an inhomogeneous medium with a rough boundary is transformed (Sec. 1) with the help of a rigorous procedure for "transferring" boundary conditions to an equivalent problem in a region with a level boundary and perturbed boundary conditions [Eq. (9) below]. A statistical averaging is applied to the resulting problem (Sec. 2). As a result, equivalent boundary conditions [conditions (25)] are found for the average field at a plane surface. These conditions incorporate the presence of a roughness. Effective-roughness operators which are involved in these boundary conditions are found as a solution of a system of operator equations [Eqs. (17)]. This system of equations can be solved by a perturbation theory based on the small height of the roughness. The corresponding asymptotic expansions [Eqs. (20)–(22)] constitute one of the primary results of this study. An effective impedance is found for a rough boundary in a lamellar medium in Sec. 3; terms quadratic in the roughness height are taken into account. The problem of the distortion of *s*- and *p*-polarized surface electromagnetic waves by a roughness is solved.

1. "TRANSFER" OF BOUNDARY CONDITIONS IN THE INITIAL PROBLEM

To avoid the particular details of practical problems,[2–5] we start from a model, as general as possible, of a medium with a nonplanar boundary. This boundary is specified by the equation $z = z_\Sigma(\mathbf{r})$. We assume that the entire three-dimensional space is divided by this boundary into two regions, V_1^r $(-\infty < z < z_\Sigma)$ and V_2^r $(z_\Sigma < z < +\infty)$. We will

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not be specific about the shape of the smooth function $z_\Sigma(\mathbf{r})$; here $\mathbf{r}=(x,y,0)$. Each of the regions $V_{1,2}^r$ is filled with an inhomogeneous and generally anisotropic medium. We ignore the nonlinearity and spatial dispersion of the medium. The problem in which we are interested can be written as follows for the complex amplitudes $\mathbf{E}(\mathbf{R})$ and $\mathbf{H}(\mathbf{R})$ of a monochromatic ($\propto e^{-i\omega t}$) electromagnetic field:

$$\nabla\times\mathbf{E}-ik_0\hat{\mu}_j\mathbf{H}=-(4\pi/c)\mathbf{M},\ \nabla\times\mathbf{H}+ik_0\hat{\varepsilon}_j\mathbf{E}=(4\pi/c)\mathbf{J},\ \mathbf{R}\in V_j^r; \quad (1)$$

$$(\mathbf{z}_0-\boldsymbol{\gamma})\times(\mathbf{E}_2-\mathbf{E}_1)=0,\quad (\mathbf{z}_0-\boldsymbol{\gamma})\times(\mathbf{H}_2-\mathbf{H}_1)=0,\quad \mathbf{R}=\mathbf{R}_\Sigma. \quad (2)$$

Here $j=1,2$; c is the velocity of light in vacuum; $k_0=\omega/c$; $\hat{\mu}_j$ and $\hat{\varepsilon}_j$ are arbitrary piecewise-smooth functions of the variable $\mathbf{R}=(x,y,z)$ which characterize the magnetodielectric properties of the medium in region V_j^r; $\mathbf{E}_j$ and $\mathbf{H}_j$ are the values of the field vectors in medium j; $\gamma=\operatorname{grad} z_\Sigma$; $\mathbf{z}_0$ is a unit vector along the z axis, which for definiteness we direct vertically upward; $\mathbf{R}_\Sigma$ is the radius vector of a point on the surface Σ ($z=z_\Sigma$); and $\mathbf{M}$ and $\mathbf{J}$ are respectively magnetic and electrical external sources. We are using dyad notation (no index) for tensor quantities. Conditions (2) require continuity of the tangential field components at the surface Σ. Corresponding conditions at other interfaces are "built into" Maxwell's equations (1) and will not be mentioned further. In situations typical of optics, the external field sources are at infinity, and the magnetic permeability is always 1: $\hat{\mu}_{1,2}\equiv 1$. However, we will formulate the problem in a more general way, so it will also apply to the rf range.[1,2,17] There are thus magnetic as well as electrical external sources in (1).

One difficulty in finding an analytic solution of this problem is that the boundary Σ does not coincide with a coordinate surface in an appropriate coordinate system. We will not specify the nature of the surface Σ, i.e., whether it is deterministic or random, since that point is unimportant in this section of the paper. We will write out a rigorous procedure which makes it possible to avoid this difficulty. The basic idea of this procedure, which stems from Refs. 1 and 12, is to couple the values of the field at spatially separate points in such a way that coupling conditions (2) at the boundary Σ are equivalent to a condition on another surface, which is more appropriate for our purposes.

It is convenient at this point to assume that the piecewise-smooth functions $\hat{\mu}_j(\mathbf{R})$ and $\hat{\varepsilon}_j(\mathbf{R})$, which are coefficients in Eq. (1), are specified throughout three-dimensional space at the outset. The overall medium specified in the formulation of the problem in (1) and (2) may be thought of as the result of "gluing" a part V_1^r of medium 1, which fills the entire space, to a part V_2^r of medium 2, which fills another copy of the space. This gluing takes place along the boundary Σ. We assume that the surface Σ (or all realizations of it, if Σ is random) can be bracketed by two planes $z=a$ and $z=b$ ($b>a$). To relax the requirements imposed as a result on the entire set of suitable planes, we choose those two planes whose separation $b-a$ is at the smallest. We assume that within the plane layer $a<z<b$ the functions $\hat{\mu}_j(\mathbf{R})$ and $\hat{\varepsilon}_j(\mathbf{R})$ depend on only z and reduce to the form

$$\hat{\mu}_j=\mu_{jt}(z)\hat{I}_t+\mu_{jz}(z)\hat{I}_z,\qquad \hat{\varepsilon}_j=\varepsilon_{jt}(z)\hat{I}_t+\varepsilon_{jz}(z)\hat{I}_z. \quad (3)$$

Here $\mu_{j\nu}$, $\varepsilon_{j\nu}$, ($j=1,2$, $\nu=t,z$) are piecewise-smooth functions of the variable z which are defined on the interval $a<z<b$, $\hat{I}_z=\mathbf{z}_0\mathbf{z}_0$, $\hat{I}_t=\hat{I}-\hat{I}_z$, and $\hat{I}$ is the unit dyad. In other words, in a certain plane slab which contains the rough boundary the media on the two sides of this boundary are lamellar and uniaxial with a vertical optic axis. The case in which the media adjacent to Σ are homogeneous and isotropic is a particular case of this model. To simplify the calculations we will also assume that there are no external field sources within the slab $a<z<b$: $\mathbf{M}=\mathbf{J}\equiv 0$.

We can then show that the field values at two vertically separated points $\mathbf{R}=(r,z)$ and $\mathbf{R}'=(\mathbf{r},\xi)$ in medium j are related by

$$\begin{aligned}\mathbf{E}_j(\mathbf{R})&=\hat{C}_{ee}^{(j)}(z,\xi)\cdot\mathbf{E}_{jt}(\mathbf{R}')+\hat{C}_{em}^{(j)}(z,\xi)\cdot\mathbf{z}_0\times\mathbf{H}_{jt}(\mathbf{R}'),\\ \mathbf{H}_j(\mathbf{R})&=\hat{C}_{me}^{(j)}(z,\xi)\cdot\mathbf{z}_0\times\mathbf{E}_{jt}(\mathbf{R}')+\hat{C}_{mm}(z,\xi)\cdot\mathbf{H}_{jt}(\mathbf{R}').\end{aligned} \quad (4)$$

Here $j=1$ and 2; the index t goes with the horizontal field components. We are using dyad ($\hat{C}_{\alpha\beta}^{(j)}$) and scalar ($G_{\varepsilon,\mu}^{(j)}$ and $\Gamma_{\varepsilon,\mu}^{(j)}$) pseudodifferential operators, which act along the variable $\mathbf{r}$ and which depend on the parameters z and ξ:

$$\begin{aligned}\hat{C}_{ee}^{(j)}(z,\xi)&=-\left[\frac{\nabla_\perp}{\varepsilon_{jt}(z)}\frac{\partial_z}{\nabla_t^2}-\frac{\mathbf{z}_0}{\varepsilon_{jz}(z)}\right]G_\varepsilon^{(j)}(z,\xi)\nabla_\perp\\ &\quad+\frac{1}{\nabla_\perp^2}\mathbf{z}_0\times\nabla_\perp\Gamma_\mu^{(j)}(z,\xi)\mathbf{z}_0\times\nabla_\perp,\\ \hat{C}_{em}^{(j)}(z,\xi)&=-\frac{1}{ik_0}\left[\frac{\nabla_\perp}{\varepsilon_{jt}(z)}\frac{\partial_z}{\nabla_\perp^2}-\frac{\mathbf{z}_0}{\varepsilon_{jz}(z)}\right]\Gamma_\varepsilon^{(j)}(z,\xi)\nabla_\perp\\ &\quad+\frac{ik_0}{\nabla_\perp^2}\mathbf{z}_0\times\nabla_\perp G_\mu^{(j)}(z,\xi)\mathbf{z}_0\times\nabla_\perp,\end{aligned} \quad (5)$$

$$\begin{gathered}\hat{C}_{ee}^{(j)}\to\hat{C}_{mm}^{(j)},\quad \hat{C}_{em}^{(j)}\to-\hat{C}_{me}^{(j)},\quad e\leftrightarrow m,\quad \varepsilon\leftrightarrow\mu;\\ G_\eta^{(j)}(z,\xi)\equiv G_\eta^{(j)}(z,\xi,\nabla_\perp^2),\quad \Gamma_\eta^{(j)}(z,\xi)\equiv\Gamma_\eta^{(j)}(z,\xi,\nabla_\perp^2),\end{gathered} \quad (6)$$

and $\nabla_\perp=\nabla-\mathbf{z}_0\partial_z$. The functions $G_\eta^{(j)}(z,\xi,\lambda)$ and $\Gamma_\eta^{(j)}(z,\xi,\lambda)$ (λ is a complex parameter; $\eta=\varepsilon,\mu$) are defined by

$$\begin{aligned}&\eta_{jt}(\xi)\Gamma_\eta^{(j)}(z,\xi,\lambda)=\partial_\xi G_\eta^{(j)}(z,\xi,\lambda),\\ &G_\eta^{(j)}(z,\xi,\lambda)\\ &\quad=[\Psi_\eta^{(j)}(z,\lambda)\Phi_\eta^{(j)}(\xi,\lambda)-\Psi_\eta^{(j)}(\xi,\lambda)\Phi_\eta^{(j)}(z,\lambda)]/\Delta_\eta^{(j)},\\ &\eta_{jt}(\xi)\Delta_\eta^{(j)}(\lambda)\\ &\quad=\Psi_\eta^{(j)}(\xi,\lambda)\partial_\xi\Phi_\eta^{(j)}(\xi,\lambda)-\Psi_\eta^{(j)}(\xi,\lambda)\partial_\xi\Phi_\eta^{(j)}(\xi,\lambda),\end{aligned} \quad (7)$$

(no summation is to be carried out over the index j). The functions $\Psi_\eta^{(j)}(z,\lambda)$ and $\Phi_\eta^{(j)}(z,\lambda)$ are defined as arbitrary linearly independent solutions of the equation

$$\left[\eta_{jz}(z)\frac{\partial}{\partial z}\frac{1}{\eta_{jt}(z)}\frac{\partial}{\partial z}+k_{\eta j}^2(z)+\lambda\right]v(z)=0 \quad (8)$$

($a<z<b$), which are continuous along with $\dot{v}/\eta_{jt}$. The dot means here the partial derivative with respect to z, $k_{\varepsilon j}^2=k_0^2\varepsilon_{jz}\mu_{jt}$, and $k_{\mu j}^2=k_0^2\mu_{jz}\varepsilon_{jt}$.

The validity of coupling equations (4) can be verified directly and easily: They become identities at $z=\xi$, while at $z\neq\xi$ the vectors $\mathbf{E}_j(\mathbf{R})$ and $\mathbf{H}_j(\mathbf{R})$ from (4) obey the homogeneous Maxwell's equations (1) for the corresponding medium. Relations (4) can be interpreted in a different way, and this alternative interpretation is of fundamental importance to the discussion below. For $a<z<z_\Sigma<\xi<b$ ($a<\xi<z_\Sigma<z<b$), Eqs. (4) taken for $j=2$ ($j=1$) express a rule for continuing the vectors $\mathbf{E}_2$, $\mathbf{H}_2$ ($\mathbf{E}_1$, $\mathbf{H}_1$) out of the region V_2^r (V_1^r) into the adjacent region between Σ and the plane $z=a$ ($z=b$), where these vectors were originally not defined. A noteworthy feature of this rule is that the "continued" vectors obey the homogeneous Maxwell's equations (1) in the region $a<z<z_\Sigma$ ($z_\Sigma<z<b$) with the same coefficients $\hat{\mu}_2$, $\hat{\varepsilon}_2$ ($\hat{\mu}_1$, $\hat{\varepsilon}_1$), as before. This assertion can be verified directly.

We will use coupling relations (4) to "transfer" boundary condition (2) from the rough surface Σ to an arbitrary fixed plane $z=d$, $a\leqslant d\leqslant b$. For this purpose, we set $j=2$, $\xi=d+0$, $z=z_\Sigma$ and then $j=1$, $\xi=d-0$, $z=z_\Sigma$ in (4). We substitute the resulting expressions for $\mathbf{E}_{2,1}(\mathbf{R}_\Sigma)$, $\mathbf{H}_{2,1}(\mathbf{R}_\Sigma)$ into condition (2). As a result we find boundary conditions at the $z=d$ plane:

$$\{\mathbf{E}_{jt}(\mathbf{R}_S)\}=\{\hat{V}_{eej}\cdot\mathbf{E}_{jt}(\mathbf{R}_S)+\hat{V}_{emj}\cdot\mathbf{H}_{jt}(\mathbf{R}_S)\},$$
$$\{\mathbf{H}_{jt}(\mathbf{R}_S)\}=\{\hat{V}_{mej}\cdot\mathbf{E}_{jt}(\mathbf{R}_S)+\hat{V}_{mmj}\cdot\mathbf{H}_{jt}(\mathbf{R}_S)\}. \qquad (9)$$

These conditions are equivalent to the original conditions (2) at the rough surface. Here and below, $\mathbf{R}_S$ is the radius vector of a point in the $z=d$ plane; $\{f(j)\}=f(2)-f(1)$ for any function $f(j)$ of the discrete argument $j=1$ and 2; and $\hat{V}_{\alpha\beta j}$ are dyad operators which act along the variable r and which depend on the discrete argument j:

$$\hat{V}_{ee2,1}=\hat{I}_t-(\hat{I}_t+\gamma\mathbf{z}_0)\cdot\hat{C}_{ee}^{(2,1)}(z_\Sigma,d),$$
$$\hat{V}_{em2,1}=-(\hat{I}_t+\gamma\mathbf{z}_0)\cdot\hat{C}_{em}^{(2,1)}(z_\Sigma,d), \qquad (10)$$
$$\hat{V}_{emj}\to\hat{V}_{mej},\quad \hat{V}_{eej}\to\hat{V}_{mmj},\quad e\leftrightarrow m,\quad j=1,\,2.$$

The function z_Σ in the operators $\hat{C}_{\alpha\beta}(z_\Sigma,d\pm0)$ is not subjected to the operator $\nabla_\perp$ which appears in them.

Maxwell's equations

$$\nabla\times\mathbf{E}-ik_0\hat{\mu}_j\cdot\mathbf{H}=-(4\pi/c)\mathbf{M},$$
$$\nabla\times\mathbf{H}+ik_0\hat{\varepsilon}_j\cdot\mathbf{E}=(4\pi/c)\mathbf{J},\quad R\in V_j^d, \qquad (11)$$

considered in each of the half-spaces V_1^d ($-\infty<z<d$) and V_2^d ($d<z<+\infty$), with the coefficients $\hat{\mu}_1$, $\hat{\varepsilon}_1$ and $\hat{\mu}_2$, $\hat{\varepsilon}_2$, respectively, along with boundary conditions (9) in the $z=d$ plane, are therefore strictly equivalent to the original problem (1), (2). If the surface Σ coincides with the $z=d$ plane, we find $\hat{V}_{\alpha\beta j}=0$ from (10), and relations (9) become the usual conditions of macroscopic electrodynamics.[17] The procedure which we are proposing for transferring the boundary condition is rigorous. A different transfer procedure, based on the formula $f(z)=\exp(h\partial_\xi)f(\xi)$, $h=z-\xi$, was developed in Refs. 1 and 2. That procedure can be used on the vectors $\mathbf{E}$ and $\mathbf{H}$ only if they are infinitely differentiable functions of the variable x. This condition is not satisfied at structural interfaces in an inhomogeneous medium.

2. BOUNDARY CONDITIONS ON THE AVERAGE FIELD

In this section and below, we are dealing with the case in which the boundary Σ undergoes random variations. We accordingly assume that the function $z_\Sigma(r)$ is a random function. As a result, the operators $\hat{V}_{\alpha\beta j}$ in (9) and also the quantities $\mathbf{E}$ and $\mathbf{H}$ are random. We will formulate the problem for the average field vectors $\langle\mathbf{E}\rangle$ and $\langle\mathbf{H}\rangle$ (where the angle brackets mean a statistical average).

The derivation of expressions for $\langle\mathbf{E}\rangle$ and $\langle\mathbf{H}\rangle$ rests on the construction of mass operators[2,6] $\hat{Q}_{\alpha\beta j}$, which act along the variable r and which have the property

$$\langle\{\hat{V}_{\alpha ej}\cdot\mathbf{E}_{jt}(\mathbf{R}_S)\rangle+\hat{V}_{\alpha mj}\cdot\langle\mathbf{H}_{jt}(\mathbf{R}_S)\}\rangle$$
$$=\{\hat{Q}_{\alpha ej}\cdot\langle\mathbf{E}_{jt}(\mathbf{R}_S)\rangle+\hat{Q}_{\alpha mj}\cdot\langle\mathbf{H}_{jt}(\mathbf{R}_S)\rangle\}, \qquad (12)$$

($\alpha=e,m$). Finding the eight mass vectors $\hat{Q}_{\alpha\beta j}$ is the central problem of this section of the paper. We will reduce this problem to one of taking the average of a system of two Lippmann-Schwinger equations,

$$\varphi_k(\mathbf{R}_S)=\varphi_k^{(0)}(\mathbf{R}_S)+\Gamma_k\cdot\{V_j\cdot\varphi_j(\mathbf{R}_S)\},\quad k=1,\,2, \qquad (13)$$

which follow from problem (9), (12). We inverted the matrices

$$\varphi_j(\mathbf{R}_S)=\begin{bmatrix}\mathbf{H}_{jt}(\mathbf{R}_S)\\ \mathbf{E}_{jt}(\mathbf{R}_S)\end{bmatrix},\quad \varphi_j^{(0)}(\mathbf{R}_S)=\begin{bmatrix}\mathbf{H}_{jt}^{(0)}(\mathbf{R}_S)\\ \mathbf{E}_{jt}^{(0)}(\mathbf{R}_S)\end{bmatrix},$$

$$V_j=\begin{bmatrix}\hat{V}_{emj} & \hat{V}_{eej}\\ \hat{V}_{mmj} & \hat{V}_{mej}\end{bmatrix},\quad \Gamma_j=\begin{bmatrix}-\hat{\Gamma}_{m}^{(j)} & \hat{\Gamma}_{me}^{(j)}\\ -\hat{\Gamma}_{em}^{(j)} & \hat{\Gamma}_{ee}^{(j)}\end{bmatrix},\quad Q_j=\begin{bmatrix}\hat{Q}_{emj} & \hat{Q}_{eej}\\ \hat{Q}_{mmj} & \hat{Q}_{mej}\end{bmatrix}, \qquad (14)$$

where $j=1,2$. The original field $\mathbf{E}^{(0)}(\mathbf{R})$, $\mathbf{H}^{(0)}(\mathbf{R})$ is found as the solution of problem (9), (11) in the absence of perturbations $\hat{V}_{\alpha\beta j}=0$. This field is expressed in terms of the external sources $\mathbf{M}\equiv\mathbf{M}(\mathbf{r},z)$, $\mathbf{J}\equiv\mathbf{J}(\mathbf{r},z)$ with the help of the Green's operators $\hat{G}_{\alpha\beta}(z,\xi)$ of the unperturbed problem. These operators act along the variable r and depend on the parameters z and ξ:

$$\begin{bmatrix}\mathbf{H}^{(0)}(\mathbf{R})\\ \mathbf{E}^{(0)}(\mathbf{R})\end{bmatrix}=\frac{4\pi}{c}\int_{-\infty}^{\infty}d\xi\begin{bmatrix}\hat{G}_{mm}(z,\xi) & \hat{G}_{me}(z,\xi)\\ \hat{G}_{em}(z,\xi) & \hat{G}_{ee}(z,\xi)\end{bmatrix}\cdot\begin{bmatrix}\mathbf{M}(\mathbf{r},\xi)\\ \mathbf{J}(\mathbf{r},\xi)\end{bmatrix}. \qquad (15)$$

We assume below that the operators $\hat{G}_{\alpha\beta}(z,\xi)$ are known. They appear in the definition of the operators $\hat{\Gamma}_{\alpha\beta}^{(j)}$, which act along the variable r:

$$\hat{\Gamma}_{\alpha\beta}^{(j)}=\lim\hat{I}_t\cdot\hat{G}_{\alpha\beta}(z,d)\times\mathbf{z}_0,\qquad V_j^d\ni z\to d. \qquad (16)$$

The transformation to a system of Lippmann-Schwinger equations in (13) makes it possible to utilize the well-known averaging technique of Refs. 2, 6, and 18. Taking an "algebraic" point of view[18] for definiteness, and omitting the intermediate calculations, we write the result:

$$Q_k=T_k-\{T_j\cdot\Gamma_j\}\cdot Q_k,\qquad k=1,\,2. \qquad (17)$$

This is a system of two independent equations for the matrices $Q_{1,2}$, which combine mass operators. The matrix T_j is given by

$$T_j=\langle(1-W)^{-1}\cdot V_j\rangle, \qquad (18)$$

where $W=\{V_j\cdot\Gamma_j\}$.

It is not possible to find a solution of rigorous equations (17) in closed form because of the difficulty in inverting the operator matrix $1-W$, as is required in the method for determining $T_{1,2}$. We will consider the case basic for our purposes: that in which all realizations of the statistically rough surface Σ differ only slightly from a reference surface $z=d$. The small quantity $z_\Sigma - d$ then arises in problem (9), (11). For convenience we assume $z_\Sigma - d = \sigma\rho(r)$, where σ is a small deterministic constant, and ρ is some random function. Under these conditions the operators $\hat{V}_{\alpha\beta j}$ can be expanded[1)] in powers of σ:

$$\hat{V}_{\alpha\beta j}=\sum_{n=1}^{+\infty}\hat{V}^{(n)}_{\alpha\beta j}. \quad (19)$$

The operators $\hat{V}^{(n)}_{\alpha\beta j}$ can be found without difficulty from (5) and (10), and we will not go through the derivation here. Under our assumptions, Eqs. (17) can be solved analytically by a perturbation theory using the small parameter σ. The resulting expressions for the mass operators are

$$\hat{Q}_{\alpha\beta j}=\sum_{n=1}^{+\infty}\hat{Q}^{(n)}_{\alpha\beta j}, \quad (20)$$

where $\hat{Q}^{(n)}_{\alpha\beta j}$ is a small term of nth order in σ. We combine the operators $\hat{Q}^{(n)}_{\alpha\beta j}$ into a matrix $Q_j^{(n)}$, and the operators $\hat{V}^{(n)}_{\alpha\beta j}$ into a matrix $V_j^{(n)}$; these matrices are similar to the matrices Q_j and V_j of (14), respectively. The quantities $Q_j^{(n)}$ are then given directly by

$$Q_j^{(1)}=\langle V_j^{(1)}\rangle, \quad (21)$$

$$Q_j^{(n)}=\Big\langle V_j^{(n)}+\sum_{q=1}^{n-1}\tilde{U}_q\cdot \tilde{V}_j^{(n-q)}\Big\rangle,\qquad n=2,3,\dots;$$

$$U_1=W^{(1)},$$

$$U_n=W^{(n)}+\sum_{q=1}^{n-1}\tilde{U}_q\cdot \tilde{W}^{(n-q)},\qquad n=2,3\dots; \quad (22)$$

where $W^{(n)}=\{V_j^{(n)}\cdot\Gamma_j\}$, and $n=1, 2,\dots$. The tilde on a quantity means the operation of extracting the fluctuational component: $\tilde{\mathscr{A}}=\mathscr{A}-\langle\mathscr{A}\rangle$. As an example, we write the operators $\hat{Q}_{\alpha\beta k}$, retaining terms quadratic in σ:

$$\hat{Q}_{emk}=-\frac{\sigma}{ik_0\varepsilon_{hz}}[k_{\varepsilon h}^2\langle\rho\rangle+\nabla_\perp\langle\rho\rangle\nabla_\perp]\cdot \mathbf{z}_0\times\hat{I}_t - \frac{\sigma^2}{2ik_0\varepsilon_{hz}}\Big[k_0^2\varepsilon_{hz}\mu_{ht}\langle\rho^2\rangle-\varepsilon_{hz}\nabla_\perp\langle\rho^2\rangle\nabla_\perp\frac{1}{\varepsilon_{hz}}\Big]\cdot\mathbf{z}_0\times\hat{I}_t$$
$$+\frac{\sigma^2}{k_0^2}\Big\langle\Big\{[k_{\varepsilon j}^2\rho+\nabla_\perp\rho\nabla_\perp]\frac{1}{\varepsilon_{jz}}\cdot\mathbf{z}_0\times\hat{\Gamma}^{(j)}_{mm}\Big\}\times[k_{\varepsilon h}^2\rho+\nabla_\perp\rho\nabla_\perp]\frac{1}{\varepsilon_{hz}}\Big\rangle\cdot\mathbf{z}_0\times\hat{I}_t, \quad (23)$$

$$\hat{Q}_{eek}=\sigma\langle\rho\gamma\rangle(\varepsilon_{ht}/\varepsilon_{hz})\nabla_\perp+\frac{\sigma^2}{2}\frac{\langle\rho^2\rangle}{k_0^2\varepsilon_{hz}\mu_{hz}}(k_{\varepsilon h}^2+\nabla_\perp\nabla_\perp)\times(k_{\mu h}^2+\mathbf{z}_0\times\nabla_\perp\mathbf{z}_0\times\nabla_\perp)+\frac{\sigma^2}{k_0^2}\Big\langle\Big\{[k_{\varepsilon j}^2\rho+\nabla_\perp\rho\nabla_\perp]\frac{1}{\varepsilon_{jz}}\cdot\mathbf{z}_0\times\hat{\Gamma}^{(j)}_{me}\Big\}\times[k_{\mu h}^2\rho+\nabla_\perp\rho\nabla_\perp]\frac{1}{\mu_{hz}}\Big\rangle\cdot\mathbf{z}_0\times\hat{I}_t$$

$$\hat{Q}_{emk}\to-\hat{Q}_{mek},\qquad \hat{Q}_{eek}\to\hat{Q}_{mmk},$$
$$\varepsilon\leftrightarrow\mu,\quad \hat{\Gamma}^{(j)}_{ee}\to\hat{\Gamma}^{(j)}_{mm},\quad \hat{\Gamma}^{(j)}_{me}\to-\hat{\Gamma}^{(j)}_{em},\quad k=1,2. \quad (24)$$

The quantities $\varepsilon_{j,k\nu}$, $\mu_{j,k\nu}$ $(\nu=t,z)$, and $k_{\eta j,k}$ $(\eta=\varepsilon,\mu)$ are taken here at the point $z=d$. Going back to our original problem [(9), (11)], we easily see that the vectors $\langle\mathbf{E}\rangle$ and $\langle\mathbf{H}\rangle$ obey the same equations—Maxwell's equations (11)—as are obeyed by the quantities which have not been averaged. Taking an average of (9), using (12), we find equivalent boundary conditions on the reference surface $z=d$:

$$\{\langle\mathbf{E}_{jt}(\mathbf{R}_S)\rangle\}=\{\hat{Q}_{eej}\cdot\langle\mathbf{E}_{jt}(\mathbf{R}_S)\rangle+\hat{Q}_{emj}\cdot\langle\mathbf{H}_{jt}(\mathbf{R}_S)\rangle\},$$
$$\{\langle\mathbf{H}_{jt}(\mathbf{R}_S)\rangle\}=\{\hat{Q}_{mej}\cdot\langle\mathbf{E}_{jt}(\mathbf{R}_S)\rangle+\hat{Q}_{mmj}\cdot\langle\mathbf{H}_{jt}(\mathbf{R}_S)\rangle\}. \quad (25)$$

The mass operators $\hat{Q}_{\alpha\beta j}$ play the role of effective-irregularity operators with respect to the average field. They reflect the contributions from the entire sequence of events of scattering by roughness features in the formation of the average field.

The possibility of describing the properties of a statistically rough boundary by means of effective parameters assigned to a level surface was demonstrated in Refs. 1 and 12 with a very simple model of a medium as the example, More-complex models were considered in Refs. 2 and 12–16. Advantages of the representation which we have found for the mass operators [(20)–(22)] are that (first) it incorporates the contribution from all possible scattering events and (second) this contribution is expanded in components of decreasing significance, in accordance with the small parameter σ, which characterizes the height of the roughness features. Expressions (23) and (24) form a correlation approximation[19,20] in the theory of scattering by a rough surface. The closest existing results ignore terms $\sim\sigma^2$, which are represented by the second terms in these expressions.

3. EXAMPLES OF THE USE OF THESE NEW RESULTS

Let us consider a particular case of the model adopted above, in which the media on the two sides of the rough surface Σ are lamellar and uniaxial, with a vertical optic axis. This model is a very important one in solid state physics[5] and other applications. We accordingly assume that Eqs. (3) characterize the parameters of the media in each of regions $V^r_{2,1}$, not solely in the transition layer $a<z<b$ which contains the realizations of the surface Σ. We restrict the discussion to the case in which the average position of Σ coincides with the reference plane $z=d$: $\langle\rho\rangle\equiv 0$. The roughness irregularities are statistically homogeneous in the broad sense of the term;[6] i.e., their correlation function $B(\mathbf{r}-\mathbf{r}')\equiv\langle\rho(\mathbf{r})\rho(\mathbf{r}')\rangle$ depends in a difference fashion on $\mathbf{r}$ and $\mathbf{r}'$. For the mass operators we will use the approximation [(23), (24)] which incorporates terms quadratic in σ. The Green's operators $\hat{G}_{\alpha\beta}(z,\xi)$ for a lamellar medium are well known.[17]

Let us calculate the equivalent impedance of a statistically rough interface from the side of medium 2. By definition, the equivalent impedance operator $\hat{L}$ satisfies the identity

$$\langle\mathbf{E}_2(\mathbf{R}_S)\rangle-\hat{L}\cdot\mathbf{z}_0\times\langle\mathbf{H}_2(\mathbf{R}_S)\rangle=0 \quad (26)$$

at the reference plane $z=d$. It is assumed here that there are no external field sources in the lower half-space, V_1^r: $\mathbf{M}=\mathbf{J}\equiv 0$. Since the procedure for constructing an equivalent impedance was described in detail in Ref. 21, we proceed immediately to the result. Under the assumptions made above, the equivalent impedance operator is characterized by the homogeneous representation

$$\hat{\mathscr{L}}(\varkappa)\equiv\exp(-i\varkappa\cdot\mathbf{r})\hat{L}\exp(i\varkappa\cdot\mathbf{r}), \tag{27}$$

which is given by

$$\hat{\mathscr{L}}(\varkappa)=(\zeta+\delta\zeta)\mathbf{nn}+(\nu+\delta\nu)\mathbf{z}_0\times\mathbf{n}\mathbf{z}_0\times\mathbf{n} +l(\mathbf{n}\mathbf{z}_0\times\mathbf{n}+\mathbf{z}_0\times\mathbf{n}\mathbf{n}). \tag{28}$$

Here $\varkappa=(\varkappa_x,\varkappa_y,0)$ is a spectral parameter, $\mathbf{n}=\varkappa/\varkappa$, ζ and ν are quantities which determine the impedance dyad of the reference plane in the absence of roughness ($\rho\equiv 0$), and $\delta\zeta$, $\delta\nu$, and l are increments ($\sim\sigma^2$) which reflect the contribution of the roughness. Here are explicit expressions for these quantities:

$$\begin{aligned}\delta\zeta(\varkappa)&={}^1/_2 i\sigma^2k_0\mathrm{B}(0)[\{\dot{\mu}_{jt}(d)\}-\zeta^2\{\dot{\varepsilon}_{jt}(d)\}]\\&+\zeta r_\varepsilon\sigma^2k_{\varepsilon2}{}^2(d)\mathrm{B}(0)[n_\varepsilon-n_\mu+(m_\varepsilon-n_\varepsilon)\varkappa^2/k_{\varepsilon2}{}^2(d)]\\&+\sigma^2\varepsilon_{2t}(d)k_0{}^2\int d\varkappa'\mathrm{B}_f[K_sL_s/\Delta_s'+(\mathbf{n}\times\mathbf{n}')^2w^2M_sN_p/\Delta_p'],\end{aligned} \tag{29}$$

$$\begin{aligned}\delta\nu(\varkappa)&={}^1/_2 i\sigma^2k_0\mathrm{B}(0)[\{\dot{\mu}_{jt}(d)\}-\nu^2\{\dot{\varepsilon}_{jt}(d)\}]\\&+\nu r_\mu\sigma^2k_{\mu2}{}^2(d)\mathrm{B}(0)[n_\varepsilon-n_\mu+(n_\mu-m_\mu)\varkappa^2/k_{\mu2}{}^2(d)]\\&-\sigma^2\varepsilon_{2t}(d)k_0{}^2\int d\varkappa'\mathrm{B}_f[w^2K_pL_p/\Delta_p'+(\mathbf{n}\times\mathbf{n}')^2M_pN_s/\Delta_s'];\\ l(\varkappa)&=\sigma^2\varepsilon_{2t}(d)k_0{}^2\int d\varkappa'\mathrm{B}_f\mathbf{z}_0\cdot\mathbf{n}'\times\mathbf{n}[K_sN_s/\Delta_s'-w^2L_pM_s/\Delta_p'];\end{aligned} \tag{30}$$

$$\begin{aligned}&\zeta(\varkappa)=i\dot{\Psi}_\varepsilon{}^-(z,\varkappa)/\Psi_\varepsilon{}^-(z,\varkappa)k_0\varepsilon_{1t}(z),\\&\nu(\varkappa)=-ik_0\mu_{1t}(z)\Psi_\mu{}^-(z,\varkappa)/\dot{\Psi}_\mu{}^-(z,\varkappa),\quad z=d-0;\\&\Delta_s(\varkappa)=\gamma_\varepsilon(\varkappa)+k_0\varepsilon_{2t}(d)\zeta(\varkappa),\quad \Delta_p(\varkappa)=\nu(\varkappa)\gamma_\mu(\varkappa)+k_0\mu_{2t}(d),\\&i\gamma_\eta(\varkappa)=\dot{\Psi}_\eta{}^+(z,\varkappa)/\Psi_\eta{}^+(z,\varkappa),\quad z=d+0,\quad \eta=\mu,\varepsilon.\end{aligned} \tag{31}$$

The quantities K_α, L_α, M_α, and N_α are given in the Appendix ($\alpha=s,p$). The integration over the variable $\varkappa'=(\varkappa_x',\varkappa_y',0)$ in these expressions is to be carried out between infinite limits; the prime indicates that the corresponding quantity depends on the variable $\varkappa'$; $\mathbf{n}'=\varkappa'/\varkappa'$; $\Delta_s'=\Delta_s(\varkappa')$, $\gamma_\varepsilon'=\gamma_\varepsilon(\varkappa')$ etc. In addition, $B_f\equiv B_f(\varkappa-\varkappa')$ and $w^2=\mu_{2t}(d)/\varepsilon_{2t}(d)$. The spectral function $B_f(\varkappa)$ is

$$\mathrm{B}_f(\varkappa)=(2\pi)^{-2}\int d\mathbf{r}\exp(i\varkappa\cdot\mathbf{r})\mathrm{B}(\mathbf{r}). \tag{32}$$

We also use the notation

$$n_\varepsilon=\frac{\varepsilon_{1t}(d)}{\varepsilon_{2t}(d)},\quad m_\varepsilon=\frac{\varepsilon_{2z}(d)}{\varepsilon_{1z}(d)},\quad r_\varepsilon=\frac{\varepsilon_{2t}(d)}{\varepsilon_{2z}(d)} \tag{33}$$

and the corresponding quantities with a subscript μ, found through the interchange $\varepsilon\leftrightarrow\mu$. The functions $\Psi_\eta^+(z,\varkappa)$ and $\Psi_\eta^-(z,\varkappa)$ ($\eta=\varepsilon,\mu$) are defined on the intervals $d<z<+\infty$ and $-\infty<z<d$, respectively. They are given as a solution of Eq. (8) with $\lambda=-\varkappa^2$, considered with $j=2$ (Ψ_η^+) or $j=1$ (Ψ_η^-) on the corresponding z interval. They must in addition satisfy the continuity condition, along with $\eta_{2t}{}^{-1}\dot{\Psi}_\eta{}^+$ and $\eta_{1t}{}^{-1}\dot{\Psi}_\eta{}^-$, and the radiation conditions Ψ_η^+ as $z\to+\infty$ and Ψ_η^- as $z\to-\infty$. It is assumed below that these functions are known.

It can be seen from these expressions that the equivalent impedance dyad $\hat{\mathscr{L}}(\varkappa)$ is determined by (in addition to the height spectrum of the roughness features) only the "external" characteristics of the reference plane, which do not depend explicitly on the particular profile of the parameters of the medium. The coefficients ν and ζ (which determine the impedance dyad in the absence of a roughness), the quantities $\gamma_{\varepsilon,\mu}$, and the values of the parameters of the media and their first derivatives with respect to z in the reference plane are external characteristics. The integration over χ' in (29), (30) incorporates the contribution to the shaping of the equivalent impedance dyad from scattering by the roughness irregularities into waves with all possible values of the wave vector $\varkappa'$. The expression in the integrand with the denominator Δ_s' (Δ_p') describes scattering into s- (p-) polarized waves. In these expressions and also in (37) and (40) below, it is assumed that these denominators do not vanish at real values of $\varkappa'$. The latter case constitutes a limiting case of our analysis. The equivalent impedance dyad which we have found becomes the result of Ref. 8 for a piecewise-homogeneous model of the medium. Expressions (29) have some terms with $\dot{\varepsilon}_{jt}$ and $\dot{\mu}_{jt}$ which are not present in the corresponding result in Ref. 21 for a lamellar medium. This result is attributed to an error of the method used there to transfer the boundary condition, which incorporates only the terms linear in σ.

We assume that there are no external sources in problem (9), (11): $\mathbf{M}=\mathbf{J}\equiv 0$. We consider the question of the natural waves of a lamellar medium with a rough boundary. Under our assumptions, this problem has a solution in the form of a spatial harmonic:

$$\begin{aligned}\langle\mathbf{E}(\mathbf{R})\rangle&=\mathbf{E}(\varkappa,z)\exp(i\varkappa\cdot\mathbf{r}),\\ \langle\mathbf{H}(\mathbf{R})\rangle&=\mathbf{H}(\varkappa,z)\exp(i\varkappa\cdot\mathbf{r}),\end{aligned} \tag{34}$$

which is along the direction of the wave vector $\varkappa=(\varkappa_x,\varkappa_y,0)$. The vector amplitudes in medium j are given by

$$\begin{aligned}\mathbf{E}(\varkappa,z)&=\mathbf{z}_0\times\mathbf{n}\mathscr{E}-\frac{1}{k_0}\left[\frac{\mathbf{z}_0\varkappa}{\varepsilon_{jz}(z)}+\frac{i\mathbf{n}}{\varepsilon_{jt}(z)}\frac{\partial}{\partial z}\right]\mathscr{H},\\ \mathbf{H}(\varkappa,z)&=\mathbf{z}_0\times\mathbf{n}\mathscr{H}+\frac{1}{k_0}\left[\frac{\mathbf{z}_0\varkappa}{\mu_{jz}(z)}+\frac{i\mathbf{n}}{\mu_{jt}(z)}\frac{\partial}{\partial z}\right]\mathscr{E}.\end{aligned} \tag{35}$$

Here $\mathbf{n}=\varkappa/\varkappa$, $\varkappa=(\varkappa_x^2+\varkappa_y^2)^{1/2}$, and the function $\mathscr{E}$ ($\mathscr{H}$) obeys Eq. (8) with $\eta=\mu$ ($\eta=\varepsilon$) for the corresponding value $j=1, 2$. We consider natural waves which convert into s-polarized waves when the roughness vanishes ($\sigma\to 0$). We seek functions $\mathscr{H}$ and $\mathscr{E}$ in a form which leads to a satisfaction of the radiation conditions as $z\to\pm\infty$:

$$\begin{aligned}\mathscr{H}(z)&=\begin{cases}\Psi_\varepsilon{}^+(z,\varkappa)/\Psi_\varepsilon{}^+(d+0,\varkappa), & z>d,\\ (1-t_1)\Psi_\varepsilon{}^-(z,\varkappa)/\Psi_\varepsilon{}^-(d-0,\varkappa), & z<d;\end{cases}\\ \mathscr{E}(z)&=\frac{k_0\mu_{2t}(d)}{\Delta_p}\begin{cases}-\chi_2\Psi_\mu{}^+(z,\varkappa)/\Psi_\mu{}^+(d+0,\varkappa), & z>d,\\ \nu_2\nu\Psi_\mu{}^-(z,\varkappa)/\Psi_\mu{}^-(d-0,\varkappa), & z<d,\end{cases}\end{aligned} \tag{36}$$

where t_1, χ_2, and ν_2 are small coefficients ($\sim\sigma^2$). The imposition of equivalent boundary conditions (9) leads to the expressions $\chi_2=l(\varkappa)$, and

$$2t_1(\varkappa)=i\zeta\sigma^2k_0B(0)\{\varepsilon_{jt}(d)\}$$
$$+r_\varepsilon\sigma^2k_{\varepsilon2}^2(d)B(0)[1-2n_\varepsilon+n_\varepsilon n_\mu$$
$$+(2n_\varepsilon-1-n_\varepsilon m_\varepsilon)\varkappa^2/k_{\varepsilon2}^2(d)]$$
$$-2\sigma^2\varepsilon_{2t}(d)q_\varepsilon k_0^2\int d\varkappa'B_f\Big[\frac{\gamma_\varepsilon'\mathbf{n}\cdot\mathbf{n}'L_s}{\Delta_s'}$$
$$+(\mathbf{n}\times\mathbf{n}')^2\frac{k_0\mu_{2t}(d)}{\Delta_p'}N_p\Big],$$
$$\nu_2=k_0\int d\varkappa'B_f\mathbf{z}_0\cdot\mathbf{n}'\times\mathbf{n}\Big[\frac{k_0\varepsilon_{2t}(d)}{\Delta_s'}L_sP_s+\frac{N_pQ_p}{\Delta_p'}\Big],$$
$$B_f\equiv B_f(\varkappa-\varkappa'),\qquad q_{\varepsilon,\mu}=1-n_{\varepsilon,\mu},\qquad (37)$$

and also to a dispersion relation in terms of the spectral parameter $\varkappa$:

$$\zeta(\varkappa)+\delta\zeta(\varkappa)+\gamma_\varepsilon(\varkappa)/k_0\varepsilon_{2t}(d)=0. \qquad (38)$$

Correspondingly, the natural waves of p polarization in a regular medium ($\sigma=0$) are described by

$$\mathscr{H}(z)=\frac{\gamma_\mu}{w^2\Delta_s}\begin{cases}-\chi_1\Psi_\varepsilon^+(z,\varkappa)/\Psi_\varepsilon^+(d+0,\varkappa), & z>d,\\ \nu_1\dfrac{k_0\mu_{2t}(d)}{k\mu_2(d)}\dfrac{\Psi_\varepsilon^-(z,\varkappa)}{\Psi_\varepsilon^-(d-0,\varkappa)}, & z<d,\end{cases}$$
$$\mathscr{E}(z)=\begin{cases}\Psi_\mu^+(z,\varkappa)/\Psi_\mu^+(d+0,\varkappa), & z>d,\\ -(1-t_2)\dfrac{\nu\gamma_\mu}{k_0\mu_{2t}(d)}\dfrac{\Psi_\mu^-(z,\varkappa)}{\Psi_\mu^-(d-0,\varkappa)}, & z<d.\end{cases} \qquad (39)$$

Boundary conditions (9) provide the following expressions for the coefficients $\chi_1=l(\varkappa)$:

$$2t_2(\varkappa)=i\nu\sigma^2k_0B(0)\{\varepsilon_{jt}(d)\}$$
$$+r_\mu\sigma^2k_{\mu2}^2(d)B(0)[1-2n_\varepsilon+n_\varepsilon n_\mu+(2m_\mu-1$$
$$-m_\mu n_\mu)\varkappa^2/k_{\mu2}^2(d)]+2\varepsilon_{2t}(d)k_0^2\int d\varkappa'B_f[k_0l_\mu\mu_{2t}(d)L_p/\Delta_p'$$
$$+(\mathbf{n}\times\mathbf{n}')^2q_\varepsilon\gamma_\varepsilon'N_s/\Delta_s'],$$
$$\nu_1(\varkappa)=k_{\mu2}(d)\int d\varkappa'B_f\mathbf{z}_0\cdot\mathbf{n}'\times\mathbf{n}\Big[\frac{k_0L_pP_p}{\Delta_p'}+\frac{1}{\mu_{2t}(d)}\frac{N_sQ_s}{\Delta_s'}\Big]. \qquad (40)$$

They also provide a dispersion relation in terms of the spectral parameter $\varkappa$:

$$\nu(\varkappa)+\delta\nu(\varkappa)+k_0\mu_{2t}(d)/\gamma_\mu(\varkappa)=0. \qquad (41)$$

Expressions for $P_{s,p}$, $Q_{s,p}$, and l_μ are given in the Appendix; $B_f\equiv B_f(\varkappa-\varkappa')$.

It follows from (35) and (36) that the magnetic field corresponding to a natural wave has a horizontal component which is transverse with respect to the propagation direction **n** and also two small ($\sim\sigma^2$) components, one longitudinal, along **n**, and one vertical. A field of this sort is of quasi-s polarization. Correspondingly, the wave described by (35) and (39) has a quasi-p polarization. These natural modes are strictly s- or p-polarized only in the case of statistically isotropic[2] roughness irregularities. For a fixed propagation direction, each solution of Eqs. (38) and (41) is shifted with respect to the corresponding solution ($\varkappa_0$) for a medium with a smooth boundary ($\sigma=0$) by corresponding complex increments

$$\delta\varkappa_0(\mathbf{n})=-k_0\varepsilon_{2t}(d)\delta\zeta(\varkappa_0\mathbf{n})/\partial_\varkappa[\gamma_\varepsilon+k_0\varepsilon_{2t}(d)\zeta]|_{\varkappa=\varkappa_0},$$
$$\delta\varkappa_0(\mathbf{n})=-\gamma_\mu(\varkappa_0)\delta\nu(\varkappa_0\mathbf{n})/\partial_\varkappa(\nu\gamma_\mu)|_{\varkappa=\varkappa_0}, \qquad (42)$$

which depend on **n**. It follows from (30) and (42) that we have $\delta\varkappa_0(\mathbf{n})=\delta\varkappa_0(-\mathbf{n})$, i.e., that the distortion of the spectrum is the same for natural waves of the same polarization which are propagating in opposite directions. The dependence of the shift $\delta\varkappa_0$ on the propagation direction disappears only in the case of statistically isotropic roughness features.

An analogous problem for a layered half-space with a rough boundary was solved in Ref. 22. The approach taken there started from boundary conditions which can be found from (9) in the approximation linear in σ, whereas in the present paper we have taken terms $\sim\sigma^2$ into account. A consequence of this approximation is a difference between the present paper and Ref. 22 in terms of $\delta\beta=W^{-1}\delta\zeta$, $\delta\alpha=w^{-1}\delta\nu$, and the terms outside the integrals in the expressions for $t_{1,2}$. Otherwise, the results of the two papers are the same.

APPENDIX

$$K_s=k_0l_\varepsilon\mu_{2t}(d)+q_\varepsilon\gamma_\varepsilon'\zeta\mathbf{nn}',\qquad K_p=k_0l_\mu\nu\varepsilon_{2t}(d)+q_\mu\gamma_\mu'\mathbf{nn}',$$
$$I_s=l_\varepsilon\mu_{2t}(d)-\varepsilon_{2t}(d)q_\varepsilon\zeta\zeta'\mathbf{nn}',\qquad M_s=k_0\zeta\varepsilon_{2t}(d)q_\varepsilon+q_\mu\gamma_\mu',$$
$$L_p=l_\mu\varepsilon_{2t}(d)\nu\nu'-\mu_{2t}(d)q_\mu\mathbf{nn}',\qquad M_p=k_0\mu_{2t}(d)q_\mu+q_\varepsilon\nu\gamma_\varepsilon',$$
$$N_s=\varepsilon_{2t}(d)q_\varepsilon\nu\zeta'-\mu_{2t}(d)q_\mu,\qquad N_p=\mu_{2t}(d)q_\mu-\varepsilon_{2t}(d)q_\varepsilon\zeta\nu',$$
$$P_s=q_\varepsilon\gamma_\varepsilon'-q_\mu\gamma_\mu,\qquad P_p=q_\mu\gamma_\mu'-q_\varepsilon\gamma_\varepsilon,$$
$$Q_s=\gamma_\varepsilon\gamma_\varepsilon'q_\varepsilon\mathbf{nn}'-k_0^2\varepsilon_{2t}(d)\mu_{2t}(d)l_\varepsilon,$$
$$Q_p=\gamma_\mu\gamma_\mu'q_\mu\mathbf{nn}-k_0^2\varepsilon_{2t}(d)\mu_{2t}(d)l_\mu,$$
$$l_\varepsilon=q_\mu\mathbf{nn}'+(m_\varepsilon-1)\varkappa\varkappa'/k_{\varepsilon2}^2(d),$$
$$l_\mu=q_\varepsilon\mathbf{nn}+(m_\mu-1)\varkappa\varkappa'/k_{\mu2}^2(d).$$

The quantities ε_{2t}, μ_{2t}, and $k_{\varepsilon,\mu2}$ are taken at the point $z=d$.

[1)] For this expansion, the functions $\hat{\varepsilon}_j(z)$ and $\mu_j(z)$ must be regular on the interval $a\leqslant z\leqslant b$; it is assumed below that this condition is met.

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