Conditions for sub-Poisson equilibrium phonon distribution in polariton systems

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It is shown that in polariton systems in thermal equilibrium a state of the photon subsystem in which the number of photons is described by a sub-Poisson distribution arises as the temperature decreases.

1. INTRODUCTION

New collective states of the electromagnetic field, whose quantum, fluctuation, and correlation properties differ substantially from the usual (random and coherent) states, have recently been predicted in quantum optics and observed experimentally.[1–3] In particular, states with a sub-Poisson distribution of the number of field quanta have been produced and a "sub-Poisson laser" has even been built.[4]

We recall that coherent light has a Poisson photon number distribution while incoherent light has a Gaussian distribution.[5,6] Distributions that are narrower than in the case of coherent light are customarily termed sub-Poisson. These "new" states of the electromagnetic field are nonequilibrium states, and they are generated by nonlinear interaction of light with the medium in the process of lasing or scattering.

It is certainly of interest to investigate the question of the existence of thermodynamically equilibrium states of Bose fields with nonstandard statistical properties. This is primarily because the mechanism of interaction of bosons of different physical nature in condensed media in many cases exhibits the same nonlinearity as the processes employed for generating sub-Poisson states in optics.

It has recently been shown that squeezing of quantum fluctuations of the amplitudes of a Bose field can be observed in the simplest model of a degenerate parametric process in a state of thermodynamic equilibrium below some temperature.[7] But the character of the distribution of the number of quanta remains super-Poisson.

In the present paper we show for the example of some simple models employed in solid-state physics that the statistical properties of a Bose field can change as the temperature decreases and we establish the condition for the appearance of a sub-Poisson distribution.

2. MODELS OF THE POLARITON TYPE

In the theory of polaritons, model problems with Hamiltonians which are bilinear in Bose operators of two types—photons and phonons—are studied:

$$H=\sum_{k,l}\{A_{kl}a_k^+a_l^+ + A_{kl}a_ka_l + B_{kl}a_k^+a_l\},$$
$$B_{kl}=B_{lk}^*,\quad A_{kl}=A_{lk},\qquad (1)$$

where a_i^+ and a_i are creation and annihilation operators and A_{ij} and B_{ij} are characteristic frequencies. This form is very general and it can be used to describe a quite wide range of phenomena in solids, including exciton-phonon interaction in molecular crystals, light-scattering by phonons, a number of problems in the theory of magnetism, and other phenomena.

A Hamiltonian of this type can be transformed by a well-known linear transformation (see, for example, Ref. 8)

$$a_k=\sum_n (u_{kn}\alpha_n+v_{kn}^*\alpha_n^+),\qquad [\alpha_m,\alpha_n^+]=\delta_{mn}$$

to a diagonal form

$$H=E_0+\sum_n E_n\alpha_n^+\alpha_n,$$

after which different thermodynamic characteristics of a system of free quasiparticles, described by the operators α_n^+ and α_n with a spectrum E_n are usually calculated. Such quasiparticles have a complicated structure. In the case of polaritons, for example, they consist of an optical phonon interacting with photons of frequency $E/\hbar$ (Ref. 9). In this case, one component of such a quasiparticle can be investigated by experimental methods, for example, with the help of Raman scattering of light,[10] which makes it possible to determine the spectral characteristics of phonons. In what follows we shall be interested in the quantum-statistical properties of the phonon subsystem, in particular, the character of the distribution of the number of phonons in polaritons which are in an equilibrium state with temperature T. For this, it is first necessary to calculate the variance of the number of quanta in different modes:

$$V_k=\langle(a_k^+a_k)^2\rangle-\langle a_k^+a_k\rangle^2,$$

where the averaging is performed over the equilibrium state of the system (1) with some temperature T:

$$\langle\ldots\rangle=\mathrm{Sp}(\ldots\rho),\qquad \rho=\prod_n\rho_n(m)|m\rangle_{nn}\langle m|,$$

$$\rho_n(m)=\frac{\langle\alpha_n^+\alpha_n\rangle^m}{(1+\langle\alpha_n^+\alpha_n\rangle)^{1+m}},\qquad \langle\alpha_n^+\alpha_n\rangle=\left[\exp\left(\frac{E_n}{k_BT}\right)-1\right]^{-1}.$$

The condition for the appearance of a sub-Poisson distribution of the number of quanta in the k th mode evidently has the form

$$V_k<\langle a_k^+a_k\rangle.\qquad (2)$$

This inequality establishes a relation between the temperature T of the system and the microscopic characteristics appearing in the Hamiltonian (1) [the interaction parameters and the characteristic frequencies A_{kl} and V_{kl}; $E_n=E_n(A,B)$]. Physically, it is natural to expect that at sufficiently high temperatures the distribution of the number of quanta should be random (Gaussian). Therefore the

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condition (2) for fixed A and B in Eq. (1) can be satisfied only by lowering the temperature. In this case the equality

$$V_k=\langle a_k^+a_k\rangle \tag{3}$$

can be viewed as an equation for determining the threshold temperature

$$T^{\text{th}}=T^{\text{th}}(A,B),$$

below which there are significant quantum fluctuations, which are not smeared by thermal noise.

3. TWO-MODE SYSTEM

For convenience, in order to avoid complicated expressions, we confine our attention to the case of one mode of the photon field interacting with a quasiresonant mode of optical phonons. The Hamiltonian of the system has the form

$$H=\omega_a a^+a+\omega_b b^+b+\varkappa(a^+b^++ba), \tag{4}$$

where $\varkappa$ is the coupling constant, and ω_a and ω_b are the frequencies of the modes a and b. We diagonalize the Hamiltonian (4) with the canonical transformation

$$a=u\alpha+v\beta^+,\quad u^2-v^2=1,$$
$$b=\mu\alpha^++\nu\beta,\quad \nu^2-\mu^2=1, \tag{5}$$

where the operators α and β satisfy the commutation relations $[\alpha,\alpha^+]=[\beta,\beta^+]=1$ and commute with one another, while the transformation parameters have the following form:

$$u=-\nu=-\left[\frac{1+(1-k^2)^{1/2}}{2(1-k^2)^{1/2}}\right]^{1/2},\quad v=-\mu=\left[\frac{1-(1-k^2)^{1/2}}{2(1-k^2)^{1/2}}\right]^{1/2}, \tag{6}$$

where

$$k=2\varkappa/(\omega_a+\omega_b). \tag{7}$$

Since the Hamiltonian (4) is stable,[11] we have $k<1$. As a result we obtain the diagonalized Hamiltonian

$$H_d=E_\alpha\alpha^+\alpha+E_\beta\beta^+\beta+E_0 \tag{8}$$

with dimensionless eigenvalues

$$\frac{E_\alpha}{\Theta}=\frac{1}{S}\left[\frac{1+k^2}{(1-k^2)^{1/2}}+\frac{\omega}{2}\right],\quad \frac{E_\beta}{\Theta}=\frac{1}{S}\left[\frac{1+k^2}{(1-k^2)^{1/2}}-\frac{\omega}{2}\right],$$
$$\frac{E_0}{\Theta}=\frac{1}{S}\left[\frac{1+k^2}{(1-k^2)^{1/2}}-1\right].$$

Here we have introduced the following notation:

$$\omega=2\frac{\omega_a-\omega_b}{\omega_a+\omega_b},\quad S=\frac{2\Theta}{\omega_a+\omega_b},\quad \Theta=k_BT. \tag{9}$$

Next we calculate the following averages:

$$\langle a^+a\rangle=v^2[n_\alpha+n_\beta+1]+n_\alpha,\quad \langle b^+b\rangle=v^2[n_\alpha+n_\beta+1]+n_\beta,$$
$$V_a=V_b=v^2(v^2+1)[2n_\alpha n_\beta+n_\alpha+n_\beta+1],$$

where

$$n_\alpha=[\exp(E_\alpha/\Theta)-1]^{-1},\quad n_\beta=[\exp(E_\beta/\Theta)-1]^{-1}. \tag{10}$$

Here the averaging is performed over the eigenvectors of the Hamiltonian (8).

A sub-Poisson distribution for the mode a results if the

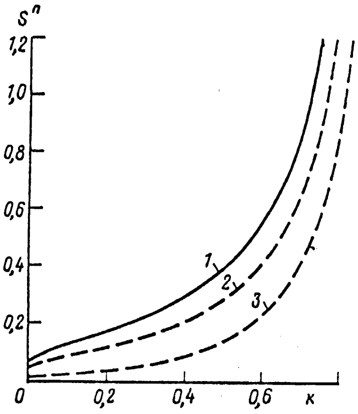

FIG. 1. Threshold temperature S^{th} [see (9) with $T=T^{\text{th}}$] versus the coupling constant k given by Eq. (7) for the following values of the detuning parameter ω: 1.0 (1), 0 (2), and -1.5 (3).

inequality (2) is satisfied. In our case this inequality has the form

$$v^4[2n_\alpha n_\beta+n_\alpha+n_\beta+1]+2n_\alpha n_\beta v^2-n_\alpha<0. \tag{11}$$

The solution of the biquadratic inequality (11) lies in the region bounded by the roots $v^2_{\pm a}$: $v^2_{-a}<v^2<v^2_{+a}$, where

$$v_{\pm a}^2=\frac{-n_\alpha n_\beta\pm d_a^{1/2}}{2n_\alpha n_\beta+n_\alpha+n_\beta+1},\quad d_a=n_\alpha^2n_\beta^2+n_\alpha[2n_\alpha n_\beta+n_\alpha+n_\beta+1].$$

Since $v^2>0$ and $v^2_{-a}<0$ hold everywhere, the restriction on v^2 will actually have the form

$$0<v^2<v_{+a}^2. \tag{12}$$

A sub-Poisson distribution for the mode b is realized when a similar inequality is satisfied:

$$0<v^2<v_{+b}^2, \tag{13}$$

where

$$v_{+b}^2=\frac{-n_\alpha n_\beta+d_b^{1/2}}{2n_\alpha n_\beta+n_\alpha+n_\beta+1}$$
$$d_b=n_\alpha^2n_\beta^2+n_\beta[2n_\alpha n_\beta+n_\alpha+n_\beta+1].$$

It is also easy to find the threshold temperature T^{th} for the mode a as a function of the parameters ω_a, ω_b, and $\varkappa$ from the equation

$$v^2(v^2+1)(2n_\alpha^{\text{th}}n_\beta^{\text{th}}+n_\alpha^{\text{th}}+n_\beta^{\text{th}}+1)$$
$$=v^2(n_\alpha^{\text{th}}+n_\beta^{\text{th}}+1)+n_\alpha^{\text{th}}. \tag{14}$$

The equation for determining the threshold temperature T^{th} for the mode b has the form

$$v^2(v^2+1)(2n_\alpha^{\text{th}}n_\beta^{\text{th}}+n_\alpha^{\text{th}}+n_\beta^{\text{th}}+1)$$
$$=v^2(n_\alpha^{\text{th}}+n_\beta^{\text{th}}+1)+n_\beta^{\text{th}}, \tag{15}$$

where n_α^{th} and n_β^{th} are determined by the expressions (10) with $T=T^{\text{th}}$. Numerical solutions of Eq. (14) are presented in Fig. 1 for different values of the dimensionless parameter ω.

4. DISCUSSION

Thus, our example of the simplest model of the polariton type shows that a sub-Poisson phonon distribution can arise at temperatures below some threshold temperature T^{th}, determined by the values of the parameters of the Hamiltonian. The condition

$$V = \langle a^{+}a\rangle$$

corresponds to a Poisson distribution, realized for the coherent state of the corresponding Bose field,[5] and this state is as close as is possible to a classical state, since it has minimum and symmetric quantum fluctuations. Hence T^{th} can be viewed as the threshold temperature of the transition from a state with significantly quantum behavior ($T < T^{\text{th}}$) into the region of classical behavior ($T > T^{\text{th}}$). Of course, a phase transition in the usual sense does not occur at the point T^{th}, since such a transition must be associated with spontaneous breaking of symmetry of the collective state of the system (see, for example, Ref. 12).

There arises the question of how this "nonclassical" behavior of phonons in thermal equilibrium can be observed experimentally. The methods of Raman scattering of light can apparently also be used for this purpose. Since, however, information about the statistical properties of phonons is contained in the second-order correlation function V, it is obviously insufficient to measure only the spectral characteristics of the scattered light. It is also necessary to measure the correlation functions of the scattered light and to reconstruct the phonon correlation function from its relation with the correlation functions of the scattered light.

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