# Maximum capacity of neutron network with four-color spins for uncorrelated images

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(Submitted 18 August 1992)
Zh. Eksp. Teor. Fiz. **103**, 1064–1069 (March 1993)

A model with $Z_N$ type of interaction is considered. The equations obtained for any $Q$ are solved for the case $Q=4$.

It was established in Ref. 1 that a system of $N$ binary spins interacting pairwise (through $N^2-N$ constants $J_{ij}$) can recognize a maximum of $2N$ images. To generalize the model of Ref. 1 we consider $Z_N$ types of interaction.

The model to be investigated is defined as follows; given $N$ complex spins taking on $Q$ possible values on the unit circle $\sigma_i=\exp(i2\pi k_i/Q)$, $k_i=1,..,Q$. The local field $H_i$ acting on the $i$th spin is defined as

$$H_i=\sum_{j\neq i}^{N}\frac{J_{ij}}{\sqrt{N}}(\sigma_i)^*, \tag{1}$$

where $J_{ij}$ are complex interaction constants, $(\sigma_i)^*$ stands for complex conjugation; the constants $J_{ij}$ are normalized to $N$, i.e.,

$$\sum_{j\neq i}^{N}|J_{ij}|^2=N. \tag{2}$$

The law of evolution of the spins $\sigma_i$ is specified as follows: in each iteration step the spin $\sigma_i$ assumes a direction such that its projection on the local field $H_i$ is a maximum. This condition, obviously, is equivalent to the requirement that the projection of $H_i$ on $\sigma_i$ be larger than the projection on two possible neighboring orientations.

A spin $\xi_i^\mu$ from an image ($\mu$ numbers the images) is thus filled if

$$\mathrm{Re}(\xi_i^\mu\sum_{j\neq i}\frac{J_{ij}}{\sqrt{N}}(\xi_j^\mu)^*)>\mathrm{Re}(\xi_i^\mu\eta\sum_{j\neq i}\frac{J_{ij}}{\sqrt{N}}(\xi_j^\mu)^*)+k \tag{3a}$$

and

$$\mathrm{Re}(\xi_i^\mu\sum_{j\neq i}\frac{J_{ij}}{\sqrt{N}}(\xi_j^\mu)^*)>\mathrm{Re}(\xi_i^\mu\eta^*\sum_{j\neq i}\frac{J_{ij}}{\sqrt{N}}(\xi_j^\mu)^*)+k, \tag{3b}$$

where $\eta=\exp(i2\varphi)$, $\varphi=\pi/Q$, $k\geqslant 0$, and $k$ is the memorization-stability parameter. Imposing the additional condition $k>0$ we obtain something by way of a "strength safety factor" for the memory, but of course at the expense of its volume.

Gardner's idea is to find that part of the phase-space volume $T$ which satisfies the conditions (2) and (3) at which it is possible to recognize $P=\alpha N$ images.

Naturally, this fraction decreases with increase of $\alpha$, since we impose on $J$ ever more conditions of type (3). For $\alpha$ larger than a certain $\alpha_c$ this volume becomes equal to zero. This $\alpha_c$ is in fact the maximum capacity of the system.

The condition (3) for remembering a number $P$ of $Q$-colored images can be written in the form

$$\prod_{\mu=1}^{P}\prod_{i=1}^{N}\theta\,(\mathrm{Re}[\xi_i^\mu H_i(1-\eta)]-k)\times\theta(\mathrm{Re}[\xi_i^\mu H_i(1-\eta^*)]-k)=1, \tag{4}$$

where $\theta(x)$ is the step function.

The fraction of the phase space (of the quantities $J_{ij}$) can be represented in the form:

$$V_T=\frac{\int\prod_{j\neq i}^{N}dJ_{ij}\,dJ_{ij}^*\prod_{\mu=1}^{P}\theta(\psi(\eta))\theta(\psi(\eta^*))\delta(\sum_{j\neq i}^{N}J_{ij}J_{ij}^*-N)}{\int\prod_{j\neq i}dJ_{ij}\,dJ_{ij}^*\delta(\sum_{j\neq i}^{N}J_{ij}J_{ij}^*-N)}, \tag{5}$$

where

$$\psi(\eta)=\mathrm{Re}[\xi_i^\mu H_i(1-\eta)]-k. \tag{6}$$

Since the choice of images is arbitrary, we must consider $\langle\ln V_T\rangle$, where $\langle\rangle$ denotes averaging over all possible configurations $P$ of the image, and ln is chosen to separate the extensive part of $V_T$. From the form of $V_T$ it follows that $\langle\ln V_T\rangle=N\langle\ln V\rangle$, where $V$ is the effective phase volume per spin [there is no summation over $i$ in (4), it is fixed, for example in this case it can be assumed that $i=1$). To calculate $\langle\ln V\rangle$ we use the replica method.

We introduce $n$ copies of our system at fixed $\{\xi_i^\mu\}$. That is to say, we consider in lieu of $J_{ij}$ $n$ copies of $J_{ij}^\alpha$, $\alpha=1...n$. The situation is the inverse of what is usually done in spin glasses, where $J_{ij}$ are fixed and $n$ copies of the spins are considered.

We obtain

$$\langle V^n\rangle=\frac{\prod_{\alpha=1}^{n}\int\prod_{j\neq i}dJ_{ij}^\alpha(dJ_{ij}^\alpha)^*\prod_{\mu=1}^{P}\theta(\psi(\eta))\theta(\psi(\eta^*))\delta(\prod_{j\neq i}^{N}J_{ij}^\alpha(J_{ij}^\alpha)^*-N)}{\prod_{\alpha=1}^{n}\int\prod_{j\neq i}dJ_{ij}^\alpha(dJ_{ij}^\alpha)^*\delta(\prod_{j\neq i}^{N}(J_{ij}^\alpha)(J_{ij}^\alpha)^*-N)}. \tag{7}$$

We use next the integral representation of the step function

$$\theta(x)=\int_0^\infty d\lambda\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\exp(i\nu(\lambda-x)). \tag{8}$$

We have to calculate

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$$\langle\prod_\alpha\theta(\psi(\eta))\theta(\psi(\eta^*))\rangle = \int_k d\lambda_\alpha^\mu d\nu_\alpha^\mu \int_{-\infty}^{\infty} \frac{dx_\alpha^\mu dy_\alpha^\mu}{(2\pi)^2}$$

$$\times\prod_\alpha \prod_{j\neq i} \langle \exp\{-i\sum_\alpha x_\alpha^\mu[\lambda_\alpha^\mu - \frac{1}{2\sqrt{N}}(\xi_i^\mu J_{ij}^\alpha(\xi_j^\mu)^*(1-\eta)$$

$$+ (\xi_i^\mu)^*(J_{ij}^\alpha)^*\xi_j^\mu(1-\eta^*)]$$

$$-i\sum_\alpha y_\alpha^\mu[\nu_\alpha^\mu - \frac{1}{2\sqrt{N}}(\xi_i^\mu J_{ij}^\alpha(\xi_j^\mu)^*(1-\eta^*)$$

$$+ (\xi_i^\mu)^*(J_{ij}^\alpha)^*\xi_j^\mu(1-\eta))]\}\rangle. \quad (9)$$

The products over $j$ can be taken from under the averaging sign in view of the separation of the variables in $i$ and $j$.

We expand next Eq. (9) in powers of $1/\sqrt{N}$ and discard all the terms of order $O(1/N)$. Averaging over $\xi$ causes all linear terms to vanish, since $\langle\xi_i\rangle = 0$, and as a result we obtain the equation

$$\langle\prod_\alpha\theta(\psi(\eta))\theta(\psi(\eta^*))\rangle = \int_k d\lambda_\alpha^\mu d\nu_\alpha^\mu \int_{-\infty}^{\infty} \frac{dx_\alpha^\mu dy_\alpha^\mu}{(2\pi)^2}\prod_{\mu,\alpha}$$

$$\times \exp[i(x_\alpha^\mu\lambda_\mu^\alpha + y_\alpha^\mu\nu_\alpha^\mu)]$$

$$\times \exp\{-(x_\alpha^{\mu 2} + y_\alpha^{\mu 2})\sin^2\varphi + 2\sin^2\varphi\cos 2\varphi\cdot x_\alpha^\mu y_\alpha^\mu$$

$$- \sum_{\alpha\neq\beta} q_{\alpha\beta}(x_\alpha^\mu x_\beta^\mu$$

$$+ y_\alpha^\mu y_\beta^\mu)\sin^2\varphi + \sum_{\alpha\neq\beta} q_{\alpha\beta}x_\alpha^\mu y_\beta^\mu\}, \quad (10)$$

Introducing the relations for $E_\alpha$ and $F_{\alpha\beta}$ for $\delta(\sum_{j\neq i}(|J_{ij}^\alpha|^2/N) - 1)$ and $\delta(\sum_{j\neq i} J_{ij}^\alpha(J_{ij}^\beta)^* - q_{\alpha\beta})$ respectively, and summing over $\alpha$, we can rewrite (10) in the form

$$\frac{\int_{-\infty}^{\infty}\prod_{\alpha=1}^n dE_\alpha dF_{\alpha\beta}dq_{\alpha\beta}/2\pi\exp\{N[\alpha G_1(q_{\alpha\beta}) + G_2(F_{\alpha\beta},E_\alpha) - \sum_{\alpha\neq\beta}F_{\alpha\beta}q_{\alpha\beta} + \sum_\alpha E_\alpha/2\}}{\int_{-\infty}^{\infty}\prod_{\alpha=1}^n dE_\alpha \exp\{N[G_2(0,E_\alpha) + \sum_\alpha E_\alpha/2]\}}, \quad (11)$$

where

$$G_1(q_{\alpha\beta}) = \ln\prod_{\alpha=1}^n \int_k d\lambda_\alpha^\mu d\nu_\alpha^\mu \int_{-\infty}^{\infty}\frac{dx_\alpha^\mu dy_\alpha^\mu}{(2\pi)^2}\prod_{\mu,\alpha}\exp[i(x_\alpha^\mu\lambda_\alpha^\mu + y_\alpha^\mu\nu_\alpha^\mu)]$$

$$\times\exp\{-\sin^2\varphi\,[x_\alpha^{\mu 2} + y_\alpha^{\mu 2} - 2\cos\varphi\cdot x_\alpha^\mu y_\alpha^\mu$$

$$+ \sum_{\alpha\neq\beta} q_{\alpha\beta}(x_\alpha^\mu x_\beta^\mu + y_\alpha^\mu y_\beta^\mu - 2\cos\varphi\cdot x_\alpha^\mu y_\beta^\mu)]\}, \quad (12)$$

$$G_2(F_{\alpha\beta},E_\alpha) = \ln\prod_{\alpha=1}^n \int dJ_\alpha dJ_\alpha^*$$

$$\times\exp\left\{-\frac{1}{2}\left[E_\alpha|J_\alpha|^2 + \sum_{\alpha\neq\beta}J_\alpha J_\beta^* F_{\alpha\beta}\right]\right\}, \quad (13)$$

where $\alpha = P/N$ is the capacity of the system.

In the limit as $N\to\infty$ we can obtain (13) by the saddle-point method with respect to the parameters $F_{\alpha\beta}$, $E_\alpha$, and $q_{\alpha\beta}$ over the function

$$G = \alpha G_1(q_{\alpha\beta}) + G_2(F_{\alpha\beta},E_\alpha) - \sum_{\alpha\neq\beta}\frac{F_{\alpha\beta}q_{\alpha\beta}}{2} + \frac{1}{2}\sum_\alpha E_\alpha. \quad (14)$$

We seek a solution in a replica-symmetric form

$$E_{\alpha\beta} = F, \qquad E_\alpha = E, \qquad q_{\alpha\beta} = q. \quad (15)$$

We use this fact and the transformation

$$\exp\left(\frac{-a^2}{2}\right) = \int_{-\infty}^{\infty} Dt\exp(iat), \qquad Dt = \frac{\exp(-t^2/2)}{\sqrt{2\pi}}dt. \quad (16)$$

We obtain

$$G_1 = \ln\prod_{\alpha=1}^n \int_{-\infty}^{\infty} DtDp\int dx_\alpha dy_\alpha d\lambda_\alpha d\nu_\alpha$$

$$\times\exp[i(x_\alpha\lambda_\alpha + y_\alpha\nu_\alpha)]\exp\{-\sin^2\varphi$$

$$\times(1-q)[\cos^2\varphi(x_\alpha - y_\alpha)^2 + \sin^2\varphi(x_\alpha + y_\alpha)^2]$$

$$+ i\sin\varphi\cos\varphi\sqrt{2q} + (x_\alpha - y_\alpha) + i\sin^2\varphi\sqrt{2q}p(x_\alpha + y_\alpha)\}. \quad (17)$$

After diagonalizing the quadratic form in (17), integrating over $(x_\alpha + y_\alpha)/2$ and $(x_\alpha - y_\alpha)/2$ and letting $n\to 0$ we obtain

$$G_1 = n\int_{-\infty}^{\infty} DtDp\ln\frac{1}{4\sin^3\varphi\cos\varphi\cdot(1-q)}$$

$$\times\int_k^{\infty} d\lambda\int_k^{\infty} d\nu\exp\left\{-\frac{1}{8\sin^2\varphi\cdot 4(1-q)}\right.$$

$$\times\left[\frac{(2\sin^2\varphi\cdot\sqrt{qt} + (\lambda+\nu)/\sqrt{2})^2}{\sin^2\varphi}\right.$$

$$\left.\left.+\frac{(2\sin\varphi\cos\varphi\cdot p + (\lambda-\nu)/\sqrt{2})^2}{\cos^2\varphi}\right]\right\}. \quad (18)$$

To take $Dt$ and $Dp$ from under the logarithm sign, we use the fact that as $n\to 0$

$$\ln\int Dtf^n(t) = \ln\int Dt[1 + n\ln f(t)] = n\int Dt\ln f(t). \quad (19)$$

It is difficult to estimate $G_1$ for arbitrary $Q$. At $Q=4$ the crossover terms of type $\lambda\nu$ in the exponential vanish and $G_1$ takes the form

$$G_1 = \alpha n \int_{-\infty}^{\infty} Dt Dp \ln \frac{1}{(1-q)} \int_k^{\infty} d\lambda d\nu \times \exp\left\{-\frac{1}{2(1-q)} [q(t^2+p^2)+\lambda^2+\nu^2 + \sqrt{2q}(t-p)+\sqrt{2q}(t+p)]\right\}. \quad (20)$$

Introducing $X=(t+p)/\sqrt{2}$ and $Y=(t-p)/\sqrt{2}$, we obtain

$$G_1 = \alpha n \int DXDY \ln H\left(\frac{\sqrt{q}X+k}{\sqrt{1-q}}\right) H\left(\frac{\sqrt{q}Y+k}{\sqrt{1-q}}\right), \quad (21)$$

where $H_x$ is the supplementary error function:

$$H(x) = \int_x^{\infty} Dz. \quad (22)$$

Using the transformation (16) and integrating over $J$, we obtain

$$G_2 = n\int DzDz^* \ln\left\{\frac{\exp[Fzz^*/2(E+F)]}{\sqrt{E+F}}\right\} = \frac{F}{2(E+F)} - \frac{\ln(E+F)}{2}. \quad (23)$$

The final expression for $G$ is

$$G = n\left[\alpha\int DXDY \ln\left\{H\left(\frac{\sqrt{q}X+k}{\sqrt{1-q}}\right) H\left(\frac{\sqrt{q}Y+k}{\sqrt{1-q}}\right)\right\} + \frac{1}{2}\ln(1-q) + \frac{q}{2(1-q)}\right]. \quad (24)$$

For the value of $q$ at the saddle point we obtain

$$q = \alpha(1-q)\int DXDY\left\{H^{-2}\left(\frac{X\sqrt{q}+k}{\sqrt{1-q}}\right)\exp\left[-\frac{(\sqrt{q}X+k)^2}{1-q}\right] + H^{-2}\left(\frac{\sqrt{q}\,Y+k}{\sqrt{1-q}}\right)\exp\left[-\frac{(\sqrt{q}Y+k)^2}{1-q}\right]\right\}. \quad (25)$$

It is clear hence that $q\to 0$ as $\alpha\to 0$. As $\alpha$ increases, an instant sets in when the value of $q$ from (25) reaches unity (further increase is impossible, since $|q|\leqslant 1$). As indicated by Gardner, it is this value which determines the critical value of $\alpha_c$ below which correct memorization of the images is impossible.

As $q\to 1$ we find, using an asymptotic expansion for $H(x)$:

$$\alpha_c = \frac{1}{2\int DX(X+k)^2\theta(X+k)}. \quad (26)$$

Letting $k\to 0$, we obtain

$$\alpha_c = 1. \quad (27)$$

The iteration rule (1)–(3) has $Z(Q)$ symmetry, one could therefore expect $\alpha_c$ to be equal to $Q$ and not to 1, but this is as follows from (27). This, however, is an illusory symmetry.

Were we to choose $H_i$ not in the form (1), but as

$$H_i = \sum_{j\neq i} \frac{J_{ij}\sigma_j}{\sqrt{N}}, \quad (28)$$

we would obtain the old equations for the mean field, the same as for the local field given by expression (1).

The choice (28) corresponds to a symmetry $Z(2)$ for even $Q$. The fact established by Gardner, that with the aid of $N^2$ real numbers it is possible to record by a simple algorithm $2N^2$ bits of information, is both beautiful and intriguing. If $Q=4$ we see that we can already record only $N^2$ 4-digit numbers ($\pm 1$, $\pm i$). When next? it is possible that if $Q=5$ a phase transition can set in (with violation of the replica symmetry).

[1]E. J. Gardner, J. Phys. A., Math. Gen. **21**, 257 (1988).

Translated by J. G. Adashko