# Effect of spatial dispersion on self-localized states of a field

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We show that allowing for spatial dispersion (nonlocal interactions) in models described in the local case by the Klein–Gordon equation changes a number of important features, including the number of self-localized states with circular symmetry, and forces the solutions to lose their smoothness as the nonlocalization parameter increases.

**1.** Since the well-known paper of Chiao *et al.*[1] written in 1964 there have been numerous studies concerned with the problem of time-independent self-localized states of nonlinear fields possessing circular symmetry.[2,3] It has been proved (see, e.g., Ref. 3) that the set of self-localized states with circular symmetry in the model with cubic nonlinearity,

$$\Delta\psi - \psi + \psi^3 = 0, \tag{1.1}$$

$$\lim_{r\to\infty}\psi = 0, \quad \lim_{r\to 0} r\psi_r = 0, \quad r^2 = x^2 + y^2, \quad (x, y) \in R^2, \tag{1.2}$$

is denumerable and can be ordered according to the number of zeros (nodes) of the function $\psi(r)$. It is also known that the transition to models with polynomial nonlinearities of higher order can lead to the appearance of a finite set of self-localized states with circular symmetry. For instance, in the model

$$\Delta\psi - \psi + \psi^3 - \alpha\psi^5 = 0 \tag{1.3}$$

the number of $N(\alpha)$ self-localized states is finite for any small but finite value of the positive parameter $\alpha$ and vanishes as $\alpha \to \alpha_c$ (see Ref. 4).

Below we show that allowing for nonlocal interactions or a complex spatial dispersion of waves leads to a decrease in the number of self-localized states, with the solutions near the symmetry center ($r\to 0$) losing their smoothness. This phenomenon is similar to the known loss of smoothness of the profile of nonlinear time-independent waves as the wave amplitude increases in the Whitham model[5] and related models.[6]

Notwithstanding the type of nonlocal interactions chosen below, the decrease in the number of self-localized states and the loss of the smoothness of solutions as the nonlocality parameter increases are, in our opinion, typical of a broad class of kernels of integral operators. Moreover, similar effects can be established in analyzing nonlocal generalizations of other models characteristic of the physics of nonlinear phenomena (e.g., in analyzing the analogous generalization of the nonlinear Schrödinger equation).

**2.** As one generalization of the problem of self-localized states of a nonlinear field to the case of nonlocal interactions (or complicated spatial dispersion) let us consider the following:

$$\psi_{tt} - \frac{\partial U}{\partial \psi} + \Delta\int dr' g(|r - r'|, \lambda)\psi(r', t) = 0, \tag{2.1}$$

$$\lim_{r\to\infty}\psi = 0, \quad \lim_{r\to 0} r\psi_r = 0, \quad r, r' \in R^2. \tag{2.2}$$

Here the function $U(\psi)$ serves as the characteristic of the nonlinear properties of the scalar field $\psi$ and the kernel $g(|r-r'|,\lambda)$ as the characteristic of the nonlocal interactions. In the case of rapidly decreasing kernels satisfying the condition $\lim_{\lambda\to 0} g(\rho,\lambda) = \delta(\rho)$, Eq. (2.1) can be considered a natural generalization of the nonlinear Klein–Gordon wave equation to the case of nonlocal interactions.

A simple example of a rapidly decreasing kernel of the integral operator in the two-dimensional case is

$$g(|r - r'|, \lambda) = \frac{1}{2\pi\lambda^2} K_0\left(\frac{\rho}{\lambda}\right), \quad \rho^2 = |r - r'|^2. \tag{2.3}$$

Here $K_0$ is the modified Bessel function of the second kind of zeroth order,[7] and $\lambda$ the characteristic nonlocalization parameter.

The integro-differential equation (2.1) with kernel (2.3) can be represented either as a system of locally coupled equations,

$$\psi_{tt} - \frac{\partial U}{\partial \psi} + \Delta q = 0, \quad \left(\Delta - \frac{1}{\lambda^2}\right) q = \psi, \tag{2.4}$$

or as a single fourth-order differential equation,

$$(1 - \lambda^2\Delta)\psi_{tt} - (1 - \lambda^2 U_{\psi\psi})\Delta\psi - U_\psi + \lambda^2(\nabla\psi)^2 U_{\psi\psi\psi} = 0. \tag{2.5}$$

Both representations become quite obvious if we allow for the fact that $K_0(\rho/\lambda)$ is the Green's function of the second equation in (2.4) for the auxiliary linear field $q$. The choice of the modified Bessel function as the nonlocal interaction kernel was hinted at by Whitham's approach,[5] who used as a simple nonlocal interaction kernel in the spatial one-dimensional case the Green's function of the equation

$$\left(\frac{d^2}{dx^2} - \frac{1}{\lambda^2}\right) q = \psi, \quad x \in R^1. \tag{2.6}$$

In the limit of $\lambda^2 \to 0$, Eq. (2.5) degenerates into the nonlinear Klein–Gordon equation, whose time-independent solutions for problem (2.2) have been studied by many researchers.[8]

For a nonlinearity of the form

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$$U(\psi) = -\frac{1}{2}\psi^2 + \frac{1}{4}\psi^4 \quad (2.7)$$

the problem of self-localized states of a field in the spatially one-dimensional case has the well-known solution

$$\psi(x) = \sqrt{2}/\cosh x, \quad (2.8)$$

which in $R^2$ corresponds to a planar self-localized layer. It is also known that in the case of circular symmetry $[\psi(r,\varphi) \equiv \psi(r)]$ and a nonlinearity of the form (2.7) there exists a denumerable set of self-localized solutions.

Let us study the effect of nonlocal interactions in the model specified by Eqs. (2.1)–(2.3) and (2.7) on the structure of a flat time-independent self-localized layer. We will see that even in this case certain characteristic features emerge. Equation (2.5) assumes the form

$$(1 + \lambda^2 - 3\lambda^2\psi^2)\psi_{xx} - \psi + \psi^3 - 6\lambda^2\psi_x^2\psi = 0 \quad (2.9)$$

and admits of the following first integral

$$(1 + \lambda^2 - 3\lambda^2\psi^2)^2\psi_x^2 = C + [1 + \lambda^2 - \frac{1}{2} \times (1 + 4\lambda^2)\psi^2 + \lambda^2\psi^4]\psi^2. \quad (2.10)$$

Here $C$ is the constant of the first integral. For self-localized planar solutions we have $C = 0$, in view of the asymptotic boundary conditions $\lim_{x\to\pm\infty} = 0$.

In the $p \equiv \psi_x$, $\psi$ phase plane the desired solution is represented by a homoclinal loop of the saddle point $O$ ($p = \psi = 0$):

$$9\lambda^2 p^2 = \frac{(\psi^2 - \psi_+^2)(\psi^2 - \psi_-^2)}{(\psi_c^2 - \psi^2)^2}\psi^2. \quad (2.11)$$

Here

$$4\lambda^2\psi_\pm^2 = 1 + 4\lambda^2 \pm \sqrt{1 - 8\lambda^2}, \qquad 3\lambda^2\psi_c^2 = 1 + \lambda^2. \quad (2.12)$$

Figure 1 depicts the behavior of the integral curves (2.10) with $0 \leqslant C \leqslant 20$ at two values of the nonlocalization parameter, $\lambda^2 \ll 1/8$ and $\lambda^2 \approx 1/8$. Note that $\psi_+^2 \to \infty$ and $\psi_-^2 \to 2$ as $\lambda \to 0$. Here the formulas (2.11) and (2.12) lead to the solution (2.8). The homoclinal loop of the saddle point $O$ exists only for $\lambda^2 < 1/8$, and the corresponding solution

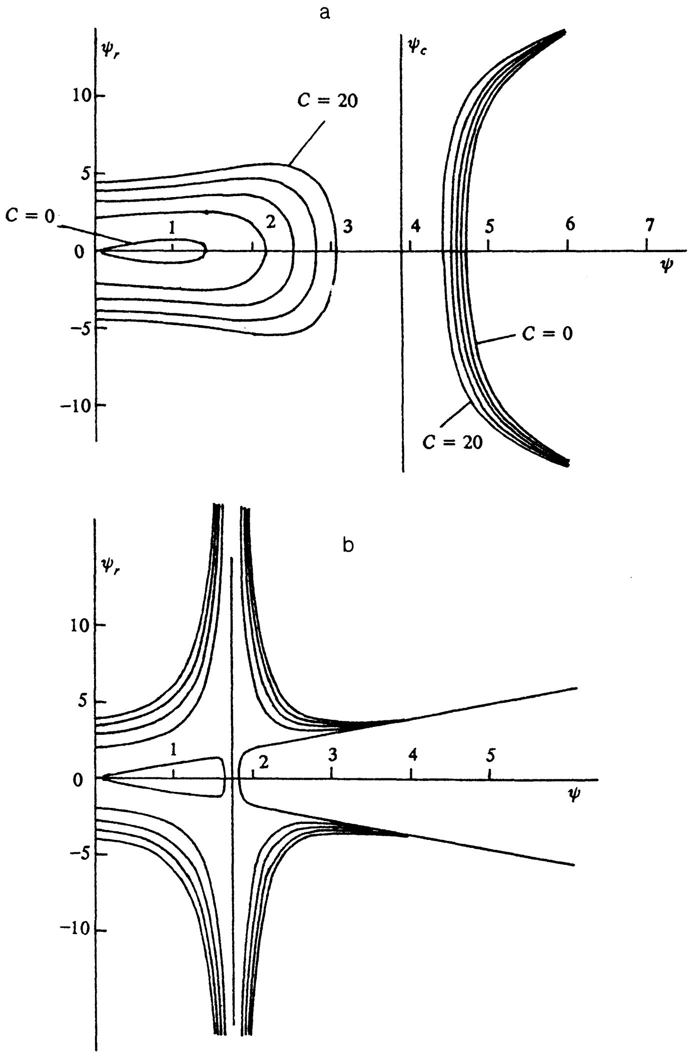


FIG. 1. Transformation of phase trajectories as the nonlocalization parameter $\lambda$ varies: (a) $\lambda^2 \ll 1/8$, and (b) $\lambda^2 \approx 1/8$.

$\psi(x,\lambda)$ loses its smoothness as $\lambda^2 \to 1/8$. Indeed, with this limiting value of the nonlocalization parameter the solution has the form

$$\psi\left(x, \lambda^2 = \frac{1}{8}\right) = \sqrt{3}\exp\left(\frac{-2\sqrt{2}}{3}|x|\right), \quad (2.13)$$

The derivative of the given solution has a kink at $x = 0$. Comparison of Figs. 1a and 1b shows the clear tendency toward the formation of this limiting solution.

When the constant of the first integral vanishes and we have $0 < \lambda^2 < 1/8$, the set of trajectories in the $p$, $\psi$ phase plane is represented by a homoclinal loop of the saddle point $O$ with $\max\psi = \psi_-$ and an open trajectory with $\min\psi = \psi_+$ and with end points going to infinity as $\psi \to \infty$. As $\lambda^2 \to 1/8$,

$$\psi_-^2 \to \psi_+^2 \to \psi_c^2 \to 3$$

and the merging of these trajectories at $\lambda^2 = 1/8$ leads to the disappearance of the loop of the saddle point $O$ for $\lambda^2 > 1/8$ (i.e., to the absence of self-localized solutions for these values of the nonlocality parameter).

Thus, allowing for nonlocal interactions in our model leads to a situation in which self-localized planar solutions cannot be continued into the range of large values of the nonlocalization parameter (the region where $\lambda^2 > 1/8$ for the case of the rapidly decreasing kernel considered above).

**3.** Let us now examine solutions self-localized in a plane and possessing circular symmetry. Expressions (2.2), (2.3), (2.5), and (2.7) in this case give rise to the following problem:

$$(1 + \lambda^2 - 3\lambda^2\psi^2)(\psi_{rr} + \frac{1}{r}\psi_r) - \psi + \psi^3 - 6\lambda^2\psi_r^2\psi = 0,$$
$$\lim_{r\to\infty}\psi = 0, \quad \lim_{r\to 0} r\psi_r = 0. \quad (3.1)$$

We set

$$\Psi = (1 + \lambda^2 - 3\lambda^2\psi^2)^2\psi_r^2 - (1 + \lambda^2)\psi^2 + \frac{1}{2}(1 + 4\lambda^2)\psi^4 - \lambda^2\psi^6, \quad (3.2)$$

which coincides with the first integral (2.10) of the problem in the case of planar geometry, and examine its variation

$$\frac{d\Psi}{dr} = -\frac{2}{r}(1 + \lambda^2 - 3\lambda^2\psi^2)^2\psi_r^2 \le 0. \quad (3.3)$$

Since $d\Psi/dr$ is nonpositive, Eq. (3.3) makes it possible to determine the general behavior of solutions to the problem (3.1) in the $p = \psi_r$, $\psi$ plane. To do so we must examine the behavior of the integral curves of problem (3.1) with respect to the family of curves $\Psi(p,\psi) \equiv \text{const}$. For a fixed value of parameter $\lambda$ the problem of self-localized slab has two families of integral curves separated by the straight line $\psi = \psi_c$, so in the case of self-localized solutions with circular symmetry the possible values of $\psi(r = 0,\lambda)$ must obey the following inequalities:

$$\psi_- < \psi(r = 0, \lambda) < \psi_c, \quad (3.4)$$

and since we have $\psi_- \to \psi_c$ as $\lambda^2 \to 1/8$, the self-localized solutions with circular symmetry disappear earlier (i.e., at smaller values of $\lambda^2$) than does a self-localized planar solution.

Numerical calculations have shown that for $0 < \lambda^2 < 1/8$ there is a finite number of self-localized solutions with circular symmetry, each of which can be continued in the nonlocalization parameter $\lambda$ to a critical value $\lambda_{cr}(n)$, where $n$ is the number of nodes of function $\psi_n(r,\lambda)$. Here the entire sequence of critical values

$$\frac{1}{8} > \lambda_{cr}^2(0) > \lambda_{cr}^2(1) > \lambda_{cr}^2(2) > \ldots > 0 \quad (3.5)$$

corresponds to the values of $\psi_n(r = 0,\lambda)$ at which the coefficient of the second derivative in Eq. (3.1) vanishes. Figure 2 depicts

$$\theta_n(\lambda) = 1 + \lambda^2 - 3\lambda^2\psi_n^2(r = 0, \lambda)$$

as a function of the nonlocalization parameter $\lambda$ for the first three self-localized solutions with circular symmetry. Examples of solutions $\psi_n(r,\lambda)$ and their images in the $p = \psi_r$, $\psi$ plane at $n = 1$, 2, and 3 are depicted, respectively, in Figs. 3, 4, and 5 for $\lambda^2 \ll \lambda_{cr}^2(n)$ and $\lambda^2 \lessdot \lambda_{cr}^2(n)$. The reader can see that, as $\lambda \to \lambda_{cr}(n)$, the first derivative of a self-localized solution develops a finite discontinuity at the symmetry center $r = 0$.

Thus, the simple model considered here suggests that allowing for nonlocal interactions severely restricts the number of self-localized states of the field, and the solutions lose their smoothness and cannot be continued in the nonlocality parameter.

**4.** We conclude this paper with a few remarks on the extent to which the spatially two-dimensional model considered above is typical.

In Ref. 9 it is shown that in both the spatially one-dimensional case and the three-dimensional case the corresponding analogs of the integro-differential equation (2.1) can be reduced to systems of auxiliary fields $q$ interacting locally with the initial nonlinear field $\psi$ for a fairly broad class of kernels $g(|r - r'|,\lambda)$. One of the conditions for such a reduction is related to the existence of direct and inverse Laplace transformations of $g(|r - r'|,\lambda)$. For instance, for the spatially three-dimensional analog of Eq. (2.1) the correponding system of equations has the form

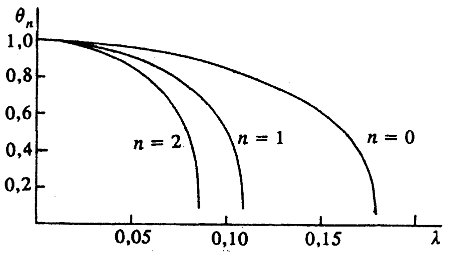


FIG. 2. The domain of definition of the first three modes; $\lambda_{cr}(0) \approx 0.180$, $\lambda_{cr}(1) \approx 0.110$, and $\lambda_{cr}(2) \approx 0.086$.

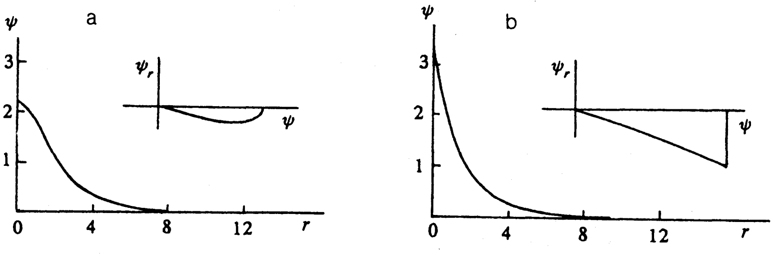


FIG. 3. Fundamental mode $\psi_0(r)$; (a) $\lambda = 0.05 \ll \lambda_{cr}(0)$ and (b) $\lambda \sim \lambda_{cr}(0)$.

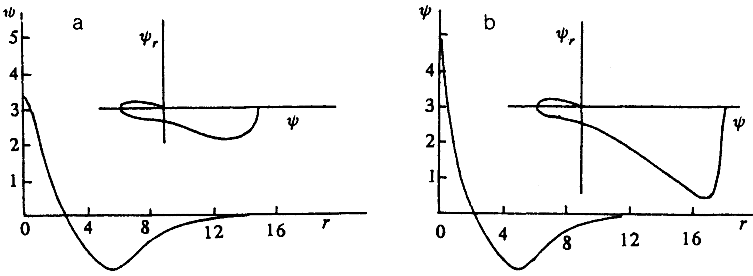


FIG. 4. First mode $\psi_1(r)$; (a) $\lambda = 0.01 \ll \lambda_{cr}(1)$ and (b) $\lambda \sim \lambda_{cr}(1)$.

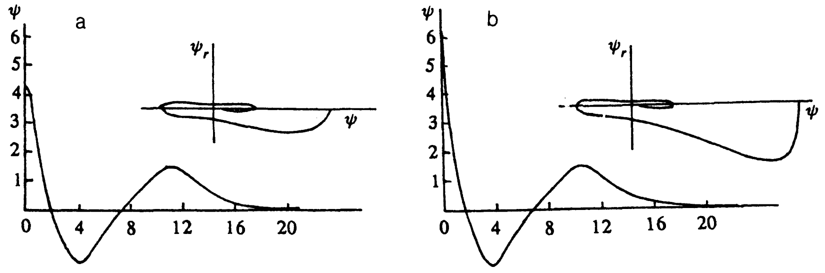


FIG. 5. Second mode $\psi_2(r)$; (a) $\lambda = 0.01 \ll \lambda_{cr}(2)$ and (b) $\lambda \sim \lambda_{cr}(2)$.

$$\psi_{tt} - \frac{\partial U}{\partial \psi} + \Delta \frac{1}{2\pi i} \int_\gamma dp \tilde{g}(p) q(r, t; p) = 0,$$

$$(\Delta - p^2) q = \psi(r, t), \qquad p \in \gamma, \qquad r \in R^3. \tag{4.1}$$

Here $\Delta$ is the three-dimensional Laplace operator, and the initial nonlocal interaction kernel $g(|r - r'|, \lambda)$ allows for direct and inverse Laplace transformations:

$$\rho g(\rho) \to \tilde{g}(p) = \int_0^\infty \rho d\rho e^{-p\rho} g(\rho),$$

$$\tilde{g}(p) \to g(\rho) = \frac{1}{2\pi i} \int_\gamma dp \frac{e^{p\rho}}{\rho} \tilde{g}(p),$$

$$\gamma \in (\sigma - i\infty, \sigma + i\infty), \qquad \rho^2 \equiv |r - r'|^2, \qquad r \in R^3. \tag{4.2}$$

The possibility of such a reduction can be related to the representation of the nonlocal interaction kernel $g(\rho)$ in the form of a Laplace integral of the function $e^{-p\rho}/\rho$, which is the Green's function of the operator $\Delta - p^2$. Here the function $\tilde{g}(\rho)$ has the meaning of the density of states of the set of the introduced auxiliary fields $q(r,t;p)$. If the kernel $g(p)$ is such that the density of states $\tilde{g}(\rho)$ has, for instance, a finite number of poles, the system of equations represents a finite number of auxiliary fields locally interacting with the initial nonlinear field $\psi(r,t)$.

Unfortunately, in the spatially two-dimensional case we do not know the direct and inverse integral transformations related to the function $K_0(p,\rho)$, which is the Green's function of the operator $\Delta - p^2$. However, there is the possibility of generalizing the above simple spatially two-dimensional model to the case where the nonlocal interaction kernel can be represented by a convolution of a certain function with the simple kernel (2.3). Specifically, suppose that

$$g(\xi, \lambda) = \frac{1}{2\pi} \int_0^\infty G(\eta, \lambda) K_0(\eta \xi) d\eta. \tag{4.3}$$

Then the nonlocal equations (2.1) can be represented as a system of locally interacting fields:

$$\psi_{tt} - \frac{\partial U}{\partial \psi} + \Delta \int_0^\infty \frac{G(\eta, \lambda)}{\eta^2} q(r, t; \eta) d\eta = 0,$$

$$(\Delta - \eta^2) q = \psi, \quad \eta \in (0, \infty). \tag{4.4}$$

Thus, the transition to a local description of the states of the initial nonlocal, nonlinear field $\psi$ is generally related to the introduction of a continuum of auxiliary linear fields $q$ whose source is the initial field. Here the function $\eta^{-2}G(\eta,\lambda)$ is interpreted as the density of states of the continuum. The assumption that the function $G(\eta,\lambda)$ has a finite number of delta function singularities also yields the finite number of auxiliary fields necessary for describing the system in terms of locally interacting fields.

With some assumptions concerning the functional behavior of the density of states this transition to a local description of the initial nonlocal model may, as shown in Ref. 6, open up new possibilities for analyzing nonlinear phenomena when the spatial dispersion (or nonlocal interactions) of the waves is taken into account.


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