Propagation of a three-dimensional optical soliton in a resonant gaseous medium

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We find the necessary conditions for the stable propagation of an ultra-short light pulse through an ensemble of two-level systems: the transverse distribution of the density of the absorbing particles must be consistent with the given transverse structure of the light beam. We discuss the case of homogeneously and inhomogeneously broadened absorption contours and the effect of the deviation of the carrier frequency of the pulse from resonance on the matching conditions for the transverse distributions of the field and of the medium. We obtain a soliton solution using perturbation theory in the small diffraction parameter, the magnitude of which is inversely proportional to the square of the radius of the light beam. The soliton solution is characterized by a shift of the carrier frequency to the red from the resonance line (there is no shift for a pulse with a plane wavefront). We consider the dynamic behavior of the propagation of a "three-dimensional pulse" in a resonant medium.

INTRODUCTION

The interest of experimental physicists in the self-induced transparency (SIT) problem has been somewhat reduced in the last ten years. This is apparently connected with the fact that sufficient theoretical and experimental data were accumulated[1–3] in the seventies to reveal all the basic features of the SIT effect. Lamb[4,5] proposed a powerful theory of this effect and showed that the Maxwell–Bloch set of equations can have soliton solutions when applied to the analysis of the coherent interaction of ultra-short light pulses with an ensemble of two-level atoms.

The recent experiments by V. S. Egorov and N. M. Reutova[6] made it necessary to look at this problem from a new point of view. Up to recently it was assumed that there were two processes limiting the optical length k_0L (k_0 is the linear absorption coefficient) a light pulse could traverse in a medium without destruction:

1. Relaxation processes, $k_0L \lesssim \tau/T_2$ where τ is the pulse length and T_2 the transverse relaxation time; in the experiments of Ref. 6 we have $\tau/T_2 \approx 3$.
2. The process of the development of a transverse instability[7,8] with $k_0L \approx 10$.

In the experiments of Ref. 6 a convergent geometry was used for the light beam: a lens was placed behind the laser light source focusing the beam onto an absorbing cell so that the focus was immediately behind the cell. Pulses passing through an absorbing medium with an optical thickness of $k_0L \approx 60$ were detected. The pulse spectrum in this case stretched beyond the limits of the Doppler absorption contour in the red direction from the resonance. The shift of the pulse carrier frequency depended in an essential way on the ratio of the density of the matter and the power of the incoming pulse; the maximum measured shift was 2400 MHz. This result made the understanding of the SIT effect much more profound.

The experiments of Ref. 6 showed that apart from the classical 2π pulse effect in a two-level medium, a pulse in a convergent beam itself produces conditions for transparency such that its carrier frequency is repelled from resonance, thereby preventing it from interacting with the medium.

The first attempts to explain this effect can be found in Refs. 9 and 10. In those it was assumed that the transverse structure of the beam remains unchanged and Gaussian during its propagation. Even in that approximation a red shift of the carrier frequency is observed away from resonance which is inversely proportional to the square of the beam radius. The fact that it is necessary to go beyond the plane-wave approximation to obtain an asymmetric frequency shift, i.e., a shift in the red direction alone turned out to be important. The shift is inversely proportional to the square of the radius of the light beam and hence does not occur in previous theoretical considerations of the SIT effect which were essentially limited to the assumption of a plane wavefront of the pulse.

More complete experimental results and a theoretical study are found in the related Refs. 11 and 12. Conditions were found in those papers for the soliton propagation of a bounded light beam in an absorbing medium which make it possible to avoid the instability mechanism described in Refs. 7 and 8. Although in the realization of the experiments of Ref. 6 no special steps were taken to ensure the conditions for the existence of a "three-dimensional soliton" one can show that the same frequency shift occurs also for the nonstationary solution which was realized in the experiment.

The aim of the present paper is a consideration of the coherent interaction of an ultra-short light pulse with a two-level medium having either a homogeneous or an inhomogeneous absorption line, taking into account the initial shift of the pulse carrier frequency away from the center of the absorption contour and taking into account the transverse structure of the field.

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BASIC EQUATIONS

We take as the starting point for our discussion of the problem the Maxwell–Bloch equations for the amplitudes of the field and the polarization assuming radial symmetry (the latter condition is necessary only to avoid complicated formulae):

$$-u_0\left[E\Delta_\perp(\varphi+\Phi)+2\frac{\partial E}{\partial\rho}\frac{\partial(\varphi+\Phi)}{\partial\rho}\right.$$

$$\left.-\frac{\partial E}{\partial v}+\frac{\sigma}{\alpha\tau}\frac{\partial E}{\partial u}=\langle Q\rangle,\right. \quad (1a)$$

$$u_0\left\{\Delta_\perp E-E\left[\frac{\partial(\varphi+\Phi)}{\partial\rho}\right]^2\right\}+\frac{\sigma}{\alpha\tau}E$$

$$\times\left[\frac{\delta\tau-\varkappa c\tau}{\sigma}-\frac{\partial\varphi}{\partial u}\right]+E\frac{\partial\varphi}{\partial v}=-\langle P\rangle, \quad (1b)$$

$$\frac{\partial P}{\partial u}=\left(\frac{\partial\varphi}{\partial u}-\Delta\omega\tau-\delta\tau\right)Q, \quad (1c)$$

$$\frac{\partial Q}{\partial u}=\left(\Delta\omega\tau+\delta\tau-\frac{\partial\varphi}{\partial u}\right)P-NE, \quad (1d)$$

$$\frac{\partial N}{\partial u}=EQ. \quad (1e)$$

In this case the complete expressions for the field and the polarization have the form:

$$\mathscr{E}(u,v,\rho)=\frac{\hbar}{d\tau}E(u,v,\rho)\exp[i\varphi(u,v,\rho)-i\Phi], \quad (2a)$$

$$\mathscr{P}(u,v,\rho)=d[P(u,v,\rho)+iQ(u,v,\rho)]$$

$$\times\exp[i\varphi(u,v,\rho)-i\Phi], \quad (2b)$$

$$\Phi=(\omega_0+\delta)t-(k+\varkappa)z. \quad (2c)$$

Here we have introduced the following notation:

$$u_0=\frac{1}{2\alpha 2kr_0^2},\quad \Delta_\perp=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho},$$

$$\alpha=\frac{2\pi kd^2n_0c\tau}{\hbar c},\quad \sigma=\frac{1}{V}-\frac{1}{c}; \quad (3)$$

V is the pulse velocity, τ is the pulse length, and d is the transition dipole moment. The angle brackets in Eqs. (1a) and (1b) denote averages over the inhomogeneously broadened line contour $g(\Delta\omega)$:

$$\langle\ldots\rangle=\int_{-\infty}^{\infty}\ldots g(\Delta\omega)d\Delta\omega.$$

In writing down Eqs. (1) we used normalized coordinates:

$$\rho=\frac{r}{r_0},\quad v=\alpha z,\quad u=\frac{t-z/V}{\tau}, \quad (4)$$

where r_0 is the radius of the light beam.

As is usual in problems in which light pulses containing a large number of optical periods propagate through not too dense resonant media, we have applied the approximation of slowly varying phases and amplitudes (a detailed analysis of this approximation was given in Ref. 13). In the equations written down above we introduced the following quantities: $E(u,v,\rho)$ and $\varphi(u,v,\rho)$, which are the slowly changing field amplitude and phase; and $P(u,v,\rho)$ and $Q(u,v,\rho)$, which are the parts of the polarization which are in phase and in quadrature with the field.

We split off in Eq. (2c) the shift of the pulse carrier frequency δ from the resonance frequency ω_0 of the two-level system and, accordingly, the correction $\varkappa$ to the wavenumber. The dispersion relation $\varkappa(\delta)$ will be found from the solution of the problem.

Equations (1a) and (1b) contain a diffraction parameter u_0 which determines the extent to which rings of different radii in the beam interact. In SIT experiments this quantity usually varies in the range from 10^{-4}–10^{-2}. The quantity u_0 shows how rapidly the change in the pulse shape along the propagation axis varies relative to the transverse changes. The smallness of the diffraction parameter u_0 justifies its choice as the perturbation theory parameter.

The Bloch equations (1c) to (1e) are written down neglecting processes involving polarization damping and inversion in the two-level system. This assumption becomes valid when the pulse length is much smaller than the relaxation times ($\tau\ll T_1$, T_2 where T_1 is the longitudinal and T_2 the transverse relaxation time). During our further discussion we take damping processes into account to first order in the time ratios τ/T_1 and τ/T_2.

Before proceeding to solve the problem we note that we do not aim at describing transition processes which precede the establishment of the self-organization solution, but concentrate our attention on looking for soliton solutions of the system (1).

ZEROTH ORDER PERTURBATION THEORY

It is difficult to find a complete analytical solution of the set of Eqs. (1). In this case it may help to choose a small parameter for the problem. We noted earlier that the diffraction parameter may claim this role because its relative smallness is guaranteed in most experimental situations.

In the zeroth order of perturbation theory in the parameter u_0 all radial derivatives are eliminated from the discussion and formally the system (1) takes the same form as in the plane-wave approximation. However, in reality all known quantities in the system (1) contain the variable ρ as a parameter and remain restricted in the transverse coordinate. We can now easily find the steady solution (i.e., the one depending solely on the wave coordinate u) for the field; it has been given several times in the literature (see, e.g., Ref. 14):

$$E_0(u,\rho)=2\ \mathrm{sech}\ u,\quad \varphi_0(u,\rho)=0. \quad (5a)$$

The remaining unknown quantities of the set of Eqs. (1) can be written down:

$$Q_0(u,\rho)=-2F(\Delta\omega)\frac{\text{sh } u}{\text{ch}^2 u},$$

$$P_0(u,\rho)=-2(\delta+\Delta\omega)F(\Delta\omega)\tau \text{ sech } u, \tag{5b}$$

$$N_0(u,\rho)=-1+2F(\Delta\omega)\text{sech}^2 u.$$

The subscript "0" indicates that we found Eqs. (5) in the zeroth order of perturbation theory. The solution (5) satisfies the following boundary conditions: the atoms are initially in their ground states, i.e., for $u=\infty$ we must have $P_0=Q_0=0$, $N_0=-1$; for any pulse we have $E_0=\partial E_0/\partial u=0$ for $u=\infty$. The parametric dependence (5) on the transverse coordinate ρ for the field, the polarization, and the inversion is contained in the characteristic parameters of the problem: $\tau(\rho)$, $V(\rho)$, $n_0(\rho)$, $\delta(\rho)$,... . Finding the actual form of those functions is the object of the exposition which follows. We find an explicit expression for the spectral response function $F(\Delta\omega)$:

$$F(\Delta\omega)=\frac{1}{1+(\Delta\omega\tau+\delta\tau)^2}. \tag{6}$$

Using (6) we can write down an expression for the velocity

$$V^{-1}=c^{-1}+\alpha_0 n_0\tau^2\langle F(\Delta\omega)\rangle, \tag{7}$$

where

$$\alpha_0=2\pi kd^2c/\hbar.$$

To complete fully the solution of the problem in the zeroth approximation we find the dispersion relation connecting the correction $\varkappa$ to the wavenumber and the shift δ of the pulse carrier frequency:

$$\varkappa=\frac{1}{c}(\delta-\alpha\tau\langle F(\Delta\omega)(\delta+\Delta\omega)\rangle). \tag{8}$$

The analysis given here is exactly the same as the solution of the problem of a pulse with a plane wavefront. Differences start when we turn to a discussion of the radial dependence of the pulse and medium parameters. It was shown in Ref. 11 for the $\delta\tau=0$ case that satisfying the matching conditions for the transverse distributions of the field and of the medium,

$$\tau^2(\rho)=\text{const}/n_0(\rho), \tag{9a}$$

guarantees that the pulse velocity V is independent of the radius ρ and it is necessary for the existence of a stationary solution for the field. This matching condition removes the cause of the pulse instability[7] whose physical mechanism is the lag of the periphery of the beam with smaller field amplitude values relative to the near-axis part where the field amplitude is larger. An analysis of Eqs. (7) and (8) generalizes the result (9a) obtained for exact resonance for a medium with a homogeneously broadened line and gives the matching condition taking into account the initial mismatch. For a homogeneously broadened absorption line ($\Delta\omega=0$) it has the form

$$n_0(\rho)\sim\tau^{-2}(\rho)+\delta^2. \tag{9b}$$

We have found the interdependence of the two medium and pulse parameters $\tau(\rho)$ and $n_0(\rho)$ as functions of the transverse coordinate. However, when there is an initial mismatch between the pulse carrier and the resonant frequency, Eq. (9b) is only one of the necessary matching conditions. We now find the second condition. In the complete expression (2a) for the field there occurs the quantity Φ defined by Eq. (2c). If the correction $\varkappa$ to the wavenumber and the mismatch δ depend on the transverse coordinate ρ the quantity Φ will also be a function of all three coordinates (t,z,ρ) and the radial derivatives of Φ in Eqs. (1a) and (1b) give a non-vanishing contribution. In order that after differentiation with respect to ρ there remain in Eqs. (1a) and (1b) only terms depending on the wave coordinate u and not containing the variable v, one must ensure that the condition

$$\delta(\rho)t-\varkappa(\rho)z=f(u,\rho) \tag{10}$$

be satisfied. We get from the form of Eq. (10) at once the factorized dependence of $f(u,\rho)$ on its variables:

$$f(u,\rho)=ug(\rho).$$

We have here introduced the functions $f(u,\rho)$ and $g(\rho)$ which are determined by the form of the initial conditions.

For a medium with a homogeneously broadened absorption line Eq. (8) can be simplified:

$$\varkappa=(c^{-1}-\alpha_0 n_0\tau^2)\delta \tag{11a}$$

or, if we use Eq. (7):

$$\varkappa=(2c^{-1}-V^{-1})\delta. \tag{11b}$$

For no realistic medium and pulse parameters are Eqs. (10) and (11b) compatible. The only possibility to retain the self-organizing feature of the solution consists in assuming the initial mismatch δ (and hence, also $\varkappa$) to be independent of the transverse coordinate ρ. Condition (10) then does not need to be satisfied since the radial derivatives of Φ in Eqs. (1a) and (1b) vanish.

We now carry out a similar study for a medium with an inhomogeneously broadened absorption line. We find several qualitative differences between the two forms of broadening.

We have already drawn attention to the fact that for steady pulse propagation one needs ensure the condition that the pulse velocity is independent of the ρ coordinate so that

$$\alpha_0 n_0(\rho)\tau^2(\rho)\left\langle\frac{1}{1+[\Delta\omega\tau(\rho)+\delta(\rho)\tau(\rho)]^2}\right\rangle=D, \tag{12}$$

where $D=\text{const}$.

If the pulse carrier frequency is out of tune with the resonance frequency one needs again to satisfy condition (10) for the existence of a self-organizing solution. Taking into account the form of the dispersion relation (8) we find the expression which connects the medium and pulse parameters:

$$\tau(\rho)\langle(2\delta(\rho)+\Delta\omega)F(\Delta\omega)\rangle=0. \tag{13}$$

To obtain simple analytical formulas from Eqs. (12) and (13) we choose for the mismatch function $g(\Delta\omega)$ a Lorentzian profile:

$$g(\Delta\omega)=\frac{T_2^*}{\pi}\frac{1}{1+(\Delta\omega T_2^*)^2}, \quad (14)$$

where T_2^* is the half-width of the Lorentzian. Equation (13) gives a relation for the parameters $\tau(\rho)$, $\delta(\rho)$, and T_2^*:

$$\delta^2(\rho)=\left(\frac{1}{T_2^*}\right)^2-\left(\frac{1}{\tau(\rho)}\right)^2. \quad (15)$$

Equation (15) determines a one-to-one connection (apart from the sign of the mismatch) between $\delta(\rho)$ and $\tau(\rho)$ assuming that the medium parameters do not vary. Equation (15) is valid if the pulse spectrum is narrower than the spectral line width; in the opposite case the self-organizing solution cannot exist. In general, the range of possible mismatch values is bounded: $0 \leqslant \delta \leqslant (T_2^*)^{-2}$.

For a function with its maximum on the axis (e.g., a Gaussian) the condition $(T_2^*)^{-2} > \tau^{-2}(\rho)$ is satisfied for all ρ if it is true on the axis ($\rho=0$).

For the same mismatch function (14) we find an analytical expression for $\langle F(\Delta\omega)\rangle$:

$$\langle F(\Delta\omega)\rangle=\frac{\tau^{-1}(\rho)[\tau^{-1}(\rho)+(T_2^*)^{-1}]}{[(T_2^*)^{-1}+\tau^{-1}(\rho)]^2+\delta^2(\rho)}. \quad (16a)$$

Taking into account the relation between the parameters expressed by Eq. (15) we can rewrite (16a):

$$\langle F(\Delta\omega)\rangle=T_2^*/2\tau(\rho). \quad (16b)$$

Equation (12) expressing one of the matching conditions takes a very simple form, if we use (16b):

$$n_0(\rho)\propto\tau^{-1}(\rho). \quad (17a)$$

All considerations given above for a medium with an inhomogeneously broadened absorption contour are valid for the case of a non-vanishing initial mismatch. If we put $\delta\tau=0$ it is no longer necessary to satisfy condition (10) and the matching condition can be expressed by a single formula:

$$n_0(\rho)\propto[(T_2^*)^{-1}+\tau(\rho)^{-1}]\tau^{-1}(\rho). \quad (17b)$$

Expression (17b) shows that in contrast to the case of a non-vanishing mismatch the density distribution depends on the spectral half-width $(T_2^*)^{-1}$ of the contour. In experiments using gaseous media the line broadening is described by a Gaussian. One can obtain the matching conditions for a function of that shape only by numerical means and we did not investigate this. At the same time it is clear from physical considerations that the various shapes of the broadening function give results similar to (16) and (17) which we considered above since they are independent of the actual details of the spectral line profile.

A comparison of the matching conditions for media with homogeneous and inhomogeneous absorption contours indicate their qualitative difference. Inhomogeneous broadening is the cause of a connection between the absolute magnitudes of the mismatch δ and the width τ^{-1} of the pulse spectrum at any distance from the axis. For homogeneous broadening τ^{-1} and δ are not connected at all.

FIRST-ORDER PERTURBATION THEORY

In the previous section we found the solution for the field amplitude and phase in the zeroth order of perturbation theory in the diffraction parameter u_0, i.e., not taking into account the effect of the mixing of rays in the beam. In the present section we shall be interested in how the form of the field changes if we take into account the mixing effect as a correction to the basic solution (5a). We consider the interaction of a light pulse with a medium with a homogeneously broadened absorption line. The first term in the square brackets in Eq. (1a) is of second order in the parameter u_0; this can be checked by looking at the solution for $\varphi_0(u,\rho)$. A nontrivial correction to the phase is realized only in first order in u_0. These considerations lead us to the conclusion that only the single term containing the radial derivatives, $\Delta_\perp E$, needs to be taken into account in first order in u_0.

The considerations given above are valid only in the case when the quantity Φ is independent of the transverse coordinate ρ.

The structure of the system of five ordinary linear differential equations which we obtain makes it possible to write down equations for the corrections to the field amplitude and phase which are not coupled to one another:

$$\frac{\partial^2 E_1(u,\rho)}{\partial u^2}+\left(\frac{6}{\mathrm{ch}^2 u}-1\right)E_1(u,\rho)$$
$$=-\delta\tau[1+(\delta\tau)^2]\Delta_\perp E_0(u,\rho), \quad (18a)$$

$$\frac{\partial^2\varphi_1(u,\rho)}{\partial u^2}-\frac{\mathrm{sh}\, u}{\mathrm{ch}\, u}\frac{\partial\varphi_1(u,\rho)}{\partial u}$$
$$=-[1+(\delta\tau)^2]\frac{\partial(\Delta_\perp E_0)}{\partial u}\frac{1}{E_0(u,\rho)}. \quad (18b)$$

One easily finds the solutions of the two equations:

$$\dot\varphi_1(u,\rho)=-(1+\delta^2\tau^2)\Big\{A(\rho)\Big[\frac{1}{2}-u\,\mathrm{th}\,u$$
$$+\frac{1}{3}\mathrm{ch}^2 u\left(\ln\mathrm{ch}\,u-\frac{\mathrm{th}^2 u}{2}-u\,\mathrm{th}^3 u\right)\Big]$$
$$+B(\rho)\Big[\frac{u^2}{2}\left(1-\frac{3}{\mathrm{ch}^2 u}\right)-2u\,\mathrm{th}\,u$$
$$+\frac{\mathrm{ch}^2 u}{3}\left(\ln\mathrm{ch}\,u-\frac{\mathrm{th}^2 u}{2}-u\,\mathrm{th}^3 u\right)\Big]$$
$$+C_1(\rho)\mathrm{ch}^2 u\Big\}, \quad (19a)$$

here we have

$$A(\rho)=\tau(\rho)\Delta_\perp \tau^{-1}(\rho),$$
$$B(\rho)=\tau^2(\rho)\left[\frac{\partial}{\partial\rho}\tau^{-1}(\rho)\right]^2. \tag{19b}$$

The solution written down for the derivative of the phase is just the expression which can be used as the basis for determining the shift of the pulse carrier frequency. The quantity $C_1(\rho)$ is determined from the boundary conditions $\varphi_1(u,\rho)E\to 0$ for $u\to\pm\infty$:

$$C_1(\rho)=[A(\rho)+B(\rho)]\left(\frac{1}{6}+\frac{\ln 2}{3}\right). \tag{19c}$$

The correction $E_1(u,\rho)$ to the field has the form

$$E_1(u,\rho)=-\delta\tau[1+(\delta\tau)^2]\Big\{A(\rho)\Big[u\Big(\ln \operatorname{ch} u-\frac{1}{3\operatorname{ch}^2 u}+\frac{5}{6}+\frac{3}{2}C_2\Big)+\frac{1}{2}\int \ln \operatorname{ch} u\,du-\frac{\operatorname{ch} u}{3\operatorname{sh} u}\ln \operatorname{ch} u+\frac{\operatorname{th} u}{6}-C_2\operatorname{cth} u+\frac{\operatorname{sh} u\operatorname{ch} u}{2}\times\Big(u\Big(\frac{2\operatorname{th}^3 u}{3}-\operatorname{th} u\Big)+C_2+\frac{1}{3}(\operatorname{th}^2 u+\ln \operatorname{ch} u)\Big)\Big]+B(\rho)\Big[-\frac{u^3}{6}+\frac{3u^2}{4}\operatorname{th}^3 u+u\Big(\frac{3}{2}\ln \operatorname{ch} u-\frac{11}{12}\operatorname{th}^2 u-\frac{1}{2}+\frac{3}{2}C_3\Big)+2\int \ln \operatorname{ch} u\,du+\frac{2}{3}\operatorname{th} u-\operatorname{cth} u\times\frac{\ln \operatorname{ch} u}{3}-C_3\operatorname{cth} u+\frac{\operatorname{sh} u\operatorname{ch} u}{2}\Big(\frac{\ln \operatorname{ch} u}{3}-\frac{u\operatorname{th}^3 u}{3}-\frac{\operatorname{th}^2 u}{6}+C_3\Big)\Big]\Big\}\frac{\operatorname{sh} u}{\operatorname{ch}^2 u}. \tag{20a}$$

The integration constants C_2 and C_3 are determined by the boundary conditions $E_1(u,\rho)\to 0$ for $u\to\pm\infty$:

$$C_2=\frac{\ln 2}{3}-\frac{1}{3},\quad C_3=\frac{\ln 2}{3}+\frac{1}{6}. \tag{20b}$$

The correction $E_1(u,\rho)$ depends on the sign of the initial mismatch $\delta\tau$ and increases in absolute magnitude when $\delta\tau$ increases. If there is no initial mismatch of the carrier frequency there is no correction to the field in first order in u_0, which agrees with the results of Ref. 11. The solution for the phase $\varphi_1(u,\rho)$ depends on $(\delta\tau)^2$ and does not vanish for $\delta\tau=0$, independent of the sign of the mismatch. We shall not enter in more detail into an analysis of the solutions (19a) and (20a); we only use expression (19a) to determine the additional shift of the pulse carrier frequency due to diffraction. To do this we average Eq. (19a) over the whole of the spectrum,[15] substituting for $E(u,\rho)$ the value $E_0(u,\rho)$:

$$\langle\dot\varphi(\rho)\rangle=\frac{\int_{-\infty}^{\infty}\Omega|\mathscr{E}(\Omega,\rho)|^2 d\Omega}{\int_{-\infty}^{\infty}|\mathscr{E}(\Omega,\rho)|^2 d\Omega}=\frac{\int_{-\infty}^{\infty}\dot\varphi E^2(u,\rho)du}{\int_{-\infty}^{\infty}E^2(u,\rho)du}, \tag{21}$$

$$\left\langle\frac{\partial\varphi(u,\rho)}{\partial t}\right\rangle=-\tfrac{1}{3}[1+(\delta\tau)^2][A(\rho)-\tfrac{1}{2}B(\rho)]\times u_0\tau^{-1}(\rho). \tag{22}$$

If we choose for $\tau^{-1}(\rho)$ the Gaussian $\tau^{-1}(\rho)=\tau_0^{-1}\exp(-\rho^2/2)$ we get for the frequency shift:

$$\left\langle\frac{\partial\varphi(u,\rho)}{\partial t}\right\rangle=\frac{2}{3}\left(1-\frac{\rho^2}{4}\right)u_0\tau^{-1}(\rho)[1+(\delta\tau)^2]. \tag{23}$$

Equation (23) determines a shift in the pulse carrier frequency in the near-axial region to the red which is independent of the sign of the initial mismatch δ. This effect was found in the experiments of V. S. Egorov and N. M. Reutova.[6]

EFFECT OF A FINITE RELAXATION TIME ON THE SOLITON PROPAGATION

In an actual experimental situation a coherent light pulse loses energy as it propagates in a resonant medium. The degree of dissipation is determined by the magnitude of the ratios τ/T_1 and τ/T_2 which we assume to be small. In Refs. 1 and 14 it is shown that if the phase of the field is zero when the interaction with the medium starts, it does not change during the further evolution. If the condition that the phase vanish is violated (we do not consider here the trivial case $\varphi=\text{const}$) the pulse carrier frequency is pulled closer to resonance or repelled from it, depending on the ratio of the times T_1 and T_2. A general feature of 2π pulses with a plane wavefront is the symmetry of their behavior relative to the center of the absorption contour: their interactions with the medium in the red and violet wings of the absorption contour are identical.

It is interesting to consider two aspects of the problem: the energy damping rate of a 2π pulse in the form of a hyperbolic secant and the shift of the pulse carrier frequency due to finite values of the longitudinal and transverse relaxation times T_1 and T_2.

We add terms P/T_2 and Q/T_2, to the left-hand sides of Eqs. (1c) and (1d), respectively, which take the polarization damping into account phenomenologically. We add to Eq. (1e) the term $(N-N_0)/T_1$ where N_0 is the equilibrium value of the difference in populations when there is no field ($N_0=-1$).

Without bothering the reader with the procedure of reaching our result we refer to Refs. 1 and 14 and write down the required expressions:

$$\frac{\partial\Gamma}{\partial v}=-2\left\langle\int_{-\infty}^{\infty}\left(\frac{Q^2+P^2}{T_2}\tau+\frac{(N+1)^2}{T_1}\tau\right)du\right\rangle. \tag{24}$$

To first order in the parameters τ/T_2 and τ/T_1 we take instead of Q, P, and N their values evaluated when there is no relaxation. The energy damping rate is then given by the equation:

$$\frac{\partial\Gamma}{\partial v}=-2\left[\frac{\langle F(\Delta\omega)\rangle}{T_2}+\frac{2}{3}\left(\frac{1}{T_1}-\frac{1}{T_2}\right)\langle F^2(\Delta\omega)\rangle\right]\tau\Gamma. \tag{25}$$

For a homogeneously broadened line, $g(\Delta\omega)=\delta(\Delta\omega)$, Eq. (25) can be simplified:

$$\frac{\partial\Gamma}{\partial v}=\frac{-2/3}{[1+(\delta\tau(\rho))^2]^2}\left\{2\frac{\tau(\rho)}{T_1}+\frac{\tau(\rho)}{T_2}\times[1+3(\delta\tau(\rho))^2]\right\}\Gamma. \tag{26}$$

If $\tau^{-1}(\rho)$ is chosen in the form of a function with a maximum on the axis (e.g., in the form of a Gaussian) the energy damping rate increases when one goes away from the axis.

We can similarly obtain the rule for the change of the pulse carrier frequency with distance:

$$\frac{\partial}{\partial v}\langle\dot{\varphi}(\rho,v)\rangle=-\frac{4}{3}\left(\frac{1}{T_1}-\frac{1}{T_2}\right)\times\langle F^2(\Delta\omega)(\delta+\Delta\omega)\rangle\tau(\rho). \tag{27}$$

Equation (27) shows that when there is no initial mismatch ($\delta\tau=0$) at the entrance into the medium the pulse does not leave in exact resonance when it propagates. If the condition $T_1^{-1}\leqslant T_2^{-1}$ is satisfied one observes that the frequency is pulled towards resonance. The rate at which it is pulled is determined not only by the values of the relaxation times, but also by the ratios of the times T_2^*, τ, and δ^{-1}. In the case of a homogeneously broadened line the maximum velocity of the frequency motion is reached when the condition $\delta\tau=1$ is satisfied.

Equations (26) and (27) are obtained, taking into account the mixing of the rays in the beam, i.e., taking into consideration terms proportional to the diffraction parameter u_0. However, their contribution to Eqs. (26) and (27) was found to be zero because in the integration over the variable u from $-\infty$ to $+\infty$ the integrands proportional to u_0 are odd.

CONCLUSION

The main result of the present study is the conclusion that three-dimensional optical solitons can exist. For this it is necessary to ensure the matching condition of the transverse distributions of the density of the two-level atoms and of the intensity of the electric field of the pulse. Also one must satisfy all the classical inequalities which ensure a self-induced transparency regime (in the near-axial region). In the experiments by V. S. Egorov and N. M. Reutova[6] a NeI gas was used as the absorbing medium and no special measures were taken to match the transverse distributions of the field and of the medium. It was just their experiments that revealed a red-shift of the pulse carrier frequency which has the same physical origin as the three-dimensional optical soliton described above.

We now turn to the physical meaning of the diffraction parameter u_0 which determines the extent to which rings of different radii interact, or, which comes to the same, how efficiently they mix. This physical picture corresponds to curvilinear propagation of rays in the beam. A problem may arise: how to reconcile the motion of each ray along a curve with the presence of a well defined radius r_0 of the beam. There is a simple explanation for this: the rays in the beam move like waves. At the leading front the ray is parallel to the axis and at the trailing front the ray is directed away from the axis, on average remaining at the same distance. This dynamic propagation of the ray is clearly illustrated by an analysis of the expression for the transverse energy flux:[16]

$$J=E^2\frac{\partial\varphi}{\partial\rho}.$$

Substituting into this Eq. (19a) for the phase we confirm the dynamic model given above: for $u>0$ we have $J>0$, while for $u<0$ we have $J<0$. For the total transverse energy flux we have $\int_{-\infty}^{\infty}dtJ=0$, so that the shape of the beam remains unaltered.

The presence of an initial mismatch δ of the pulse carrier frequency does not change the qualitative form of the phase $\varphi(u,\rho)$ of the field, but only deepens the phase modulation and increases the red-shift, independent of the sign of δ. When the mismatch is not equal to zero a correction $E_1(u,\rho)$ to the field amplitude appears which is proportional to the parameter u_0. In the case of exact resonance the correction to the field appeared only to second order in the diffraction parameter.

The effect of finite relaxation times for the polarization and the inversion on the motion of the pulse frequency is completely symmetric for the red and violet wings of the absorption contour. Under the influence of relaxation processes the pulse carrier frequency is pulled towards resonance and the efficiency (26) of the damping is increased even more.

This well known result for the plane wave case is also true for a three-dimensional optical soliton, with the one difference that at different distances from the axis the carrier frequency moves with different speeds (due to the differences in the length τ for different values of ρ). If the initial mismatch $\delta\tau$ is chosen positive, the additional phase shift (19a) accelerates the process of attraction towards resonance, increasing the rate of energy absorption (26) of the pulse. If, on the other hand, we have $\delta\tau<0$ the additional phase shift slows down the motion towards resonance and, hence, the pulse energy is less efficiently damped.

It is probable that this physical picture is the basis for an explanation of the experiments by Diels and Hahn.[17] They found that if the pulse carrier frequency is shifted away from resonance toward the violet its energy is absorbed more strongly by an order of magnitude than when the frequency is shifted by the same amount from the line center toward the red.

In conclusion we consider the possibility of realizing experimentally a three-dimensional optical soliton. To do this we must distribute resonant atoms in a transverse direction according to a law determined by the radial structure of the field. Most experience in this field has been accumulated in fiber optics where it is possible to produce in practice any profile of the refractive index. In the same way one can introduce in a waveguide additional resonant atoms with a density distribution in the transverse cross-section which is given beforehand.

A waveguide with additions of resonant atoms may find applications for obtaining pulses with given characteristics in coupling systems. Pulses propagating in a waveguide with a specially chosen transverse distribution of resonant atoms will not be subject to the instability mechanism of Ref. 7. In such a statement of the problem one must take into account the nonlinearity of the interaction of the pulse field with nonresonant atoms occupying the waveguide.

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