Solitary gravitational waves with vorticity structure at the surface of a deep fluid

V. P. Lukomskii and Yu. V. Sedletskii

Institute of Physics, Ukrainian Academy of Sciences, 252650 Kiev, Ukraine
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Within the framework of a weakly nonlinear theory, we investigate the local vortex flow of an ideal gravitating fluid with a free surface in a spatially two-dimensional environment. We obtain a model nonlinear integrodifferential equation for the complex potential of an infinitely deep fluid and find new types of steady-state flows whose velocity fields have a multipole structure near the free surface. The corresponding exact solutions to the model equation are solitary gravitational waves with vortex lines oriented transverse to the motion, and are significantly nonmonotonic in the variation of amplitude with depth. We propose a new model to explain the formation of various anomalous states of the ocean surface.

1. INTRODUCTION

Starting with the classic work of Stokes, periodic finite-amplitude gravitational waves at the surface of an ideal deep fluid have been investigated primarily within the framework of potential theory.[1] Fundamentally new nonlinear regimes of propagation of surface waves were identified after the discovery of their modulation instability against long-wavelength perturbations.[2,3] In particular, it has been proved within the framework of a weakly nonlinear theory that the asymptotically stable finite-amplitude states of a fluid are the so-called envelope solitons, i.e., modulated trains of waves propagating with a velocity close to the group velocity of the linear occupation wave. This result has stimulated further studies of the nonlinear dynamics beyond the instability threshold, along with a search for supercritical steady-state regimes.[4] Studies carried out within the framework of the Fourier spectral method, and also the results of Longuet-Higgins' numerical solution of the exact nonlinear equations of hydrodynamics,[5] have revealed the existence of a restabilization of wave trains, even those with rather high amplitudes. Bifurcation of Stokes waves in the one-dimensional case was first observed in numerical modeling by Chen and Saffman,[6] who showed that a steady-state modulated structure can exist with N waves per modulation period. Using the exact equations for potential-flow waves on deep water, they found solutions for $N = 2$ and $N = 3$, which correspond to bifurcations with period doubling and tripling. For all these cases, the observed behavior is typical of surface potential waves, with a characteristic exponential decay of the amplitude with depth.

Until very recently, a search has been underway to obtain solitary gravitational finite-amplitude waves for deep water within the framework of potential theory by a limiting transition from solutions to the problem of gravitational-capillary waves.[7,8] The existence of gravitational-capillary waves, in turn, has been verified both by numerical results and by the existence of an exact singular solution, found by Crapper for pure capillary waves.[9,10] Nevertheless, the question of whether stationary solitary gravity waves can exist on deep water remains one of the unsolved problems of the nonlinear theory.

It is also known that motions can arise in fluids of a completely different nature, i.e., motions that involve vorticity or local vorticity.[1,11] Historically, the first exact solutions to the nonlinear equations of hydrodynamics for an ideal gravitating fluid with a free surface were the Gerstner surface waves, whose vorticity is continuously distributed and exponentially decaying into the fluid.[1] Theorems on the existence of a wider class of periodic vortex waves of this kind were established later by Dubreuil-Jacotain and Guyon,[1] and for solitary waves on a fluid layer of finite depth by Moiseev and Ter-Krikhorov.[12,13] The simplest example of local vorticity waves is the Karman vortex street, which appears as a stationary structure behind a body moving in a fluid. This structure forms as a result of separation of a boundary layer and the evolving instability of a tangential discontinuity.[11,14] For this special case, the processes by which the instability develops have been investigated experimentally with considerable thoroughness; these processes are found to lead to the formation of large-scale isolated regions of nonzero vorticity throughout the volume occupied by the fluid. Thus, waves of this kind, which arise as a result of the exponential growth of the amplitude, develop subsequently into a periodic series of compact vortices oriented transverse to the flow. The regions with circulation, in which most of the vorticity is concentrated, are usually called Kelvin cat's-eyes. The experimental data show that the rolling-up of a vortex comes to a halt when the amplitude of the fundamental harmonic is sufficient for parametric excitation of a subharmonic at half the frequency.[15–18]

Thus, the evolution of an instability in the supercritical regime can lead to formation of large-scale localized vortex structures over a broad region on both sides of a bounding surface. The intensity and spatial scales of these structures determine the profile of this surface, while the presence of the gravitational force determines its dynamic properties. We might expect that under certain conditions it is possible for the surface moving in a gravitational field to act back on the vortex structure, thereby stabilizing their combined motion

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and making it asymptotically stable. It is the existence of just this type of solitary gravitational wave on deep water with trapped vortices that will be discussed in this paper.

2. STATEMENT OF THE PROBLEM AND MODEL EQUATIONS FOR A WEAKLY NONLINEAR THEORY

We will investigate local vortex flows in a uniform gravitational field on an infinitely deep ideal fluid with a free surface, all within the framework of the planar problem of hydrodynamics. Let us choose a Cartesian system of coordinates that such the y axis is directed vertically upward and the unperturbed free surface of the fluid coincides with the plane $y=0$. While a free gravitational wave is propagating, the surface profile will be defined by the equation $y=\eta(x,t)$. The flow is assumed to be potential flow except for a finite number of isolated singular points, as a result of which the region of potential flow D^- is, in general, multiply connected. Let us introduce the velocity potential $\mathbf{v}=\nabla\phi$; then the equation for the field becomes the Laplace equation

$$\phi_{xx}+\phi_{yy}=0, \quad x,y\in D^-, \quad \frac{\partial\phi}{\partial y}\Big|_{y=-\infty}=0. \tag{1}$$

The time dependence of the potential is parametric. Boundary conditions at the unknown free surface are obtained from the requirement that the hydrodynamic pressure of the moving fluid $p(x,\eta,t)$ equal the constant atmospheric pressure p_0 at all points on the surface and all times. Neglecting the density of air compared to the density of water, this equation implies that all pressure changes in the liquid are tracked instantaneously by the atmosphere. We obtain the first so-called dynamical boundary condition by applying the Cauchy–Lagrange integral at points on the free surface

$$\frac{p_0-p(x,y,t)}{\rho}\equiv\phi_t+\frac{1}{2}({\phi_x}^2+{\phi_y}^2)+gy=0,$$

$$y=\eta(x,t), \tag{2}$$

where ρ is the density and g is the acceleration in free fall. The second boundary condition is kinematic in nature: because the flow is assumed to be continuous, it can be derived from the requirement that particles of the fluid located on the moving surface at a certain time should not leave it at later times. Let us write this condition in such a form that it does not contain the unknown function $\eta(x,t)$ explicitly. For this we will consider the expression $p(x,\eta,t)-p_0=0$ to be an implicit equation for the free surface, for which $\eta(x,t)-y=0$ is an alternative equation. Then by setting the total derivatives of the left sides of these equations to zero, taking (2) into account and the definition $dy/dt=\phi_y$, we obtain

$$\phi_{tt}+g\phi_y+(v^2)_t+\tfrac{1}{2}\mathbf{v}\nabla v^2=0, \quad y=\eta(x,t). \tag{3}$$

From potential theory it is known[19] that the dimensionality of the boundary condition (1), (3), which is expressed in terms of the independent variables, can be reduced by one by introducing the boundary value of the potential. The equation for this boundary value, in the present case, is boundary condition (3). The expression for the normal derivative ϕ_y entering into (3) is easily obtained only for $y=0$. On the other hand, within the framework of the weak-nonlinearity approximation, and assuming that the singular points of the flow are located in the lower half-space at some distance from the free surface, we can reformulate (1), (3) as a boundary value problem for the half-space by displacing the boundary condition (3) to the line $y=0$. Let us introduce dimensionless variables and define a criterion for weak nonlinearity. Let l be a characteristic parameter with dimensions of length. Using the two parameters g and l we can form the following characteristic values of the velocity, potential, and time:

$$v_l=\sqrt{gl}, \quad \phi_l=lv_l, \quad t_l=l/v_l \tag{4}$$

we then use these quantities to normalize the corresponding dimensional quantities of the problem. We will retain the previous notation for the new dimensionless variables. In this case, the dimensionless equations we obtained before retain their previous form if we set $g=1$. The motion of the fluid will be weakly nonlinear if the following inequalities hold at every point (in dimensionless variables) near the free surface:

$$|\eta|<<1, \quad |v|<<1. \tag{5}$$

In this case the boundary conditions (2), (3) may be displaced from the unknown surface $y=\eta(x,t)$ to the plane $y=0$ by expanding $\phi(x,y,t)$ and its derivatives in a power series in the usual way at the point $y=0$:

$$\phi_{x^l;t^m}(x,y,t)=\phi_{x^1;t^m}(x,0,t)+\sum_{n=1}^{\infty}\frac{\partial^n\phi_{x^1;t^m}}{\partial y^n}\Big|_{y=0}\frac{y^n}{n!}, \quad l,m=0,1,\ldots. \tag{6}$$

Substituting the expansion (6) into (3) and saving terms up to cubic in the amplitude, we obtain a new boundary condition at $y=0$ (Ref. 20):

$$\varphi^{(1)}+\varphi^{(2)}+\varphi^{(3)}+\ldots=0, \tag{7}$$

$$\varphi^{(1)}=\varphi_{tt}+\varphi_y, \quad \varphi^{(2)}=(\mathbf{v}^2)_t-\varphi_t(\varphi_{tt}+\varphi_y)_y,$$

$$\varphi^{(3)}=\tfrac{1}{2}\mathbf{v}\nabla(\mathbf{v}^2)-\varphi_t(v^2)_{ty}-\tfrac{1}{2}\mathbf{v}^2\varphi_y^{(1)}+\tfrac{1}{2}(\varphi_t^2\varphi_y^{(1)})_y.$$

The notation φ rather than ϕ will be used henceforth for the potential and its derivatives at $y=0$. The Cauchy–Lagrange integral (2) can be used to find the profile of the free surface $\eta(x,t)$. To the same order of perturbation theory we obtain by the method of successive approximations

$$\eta(x,t)=-\varphi_t+\varphi_t\varphi_{ty}-\tfrac{1}{2}(\varphi_x^2+\varphi_y^2)+\eta^{(3)}(x,t)+\ldots,$$

$$\eta^{(3)}(x,t)=\tfrac{1}{2}[(\varphi_t\mathbf{v}^2)_y-\tfrac{1}{3}(\varphi_t^3)_{yy}]. \tag{8}$$

Thus, in the weakly nonlinear approximation (5) the boundary value problem (1), (3) for the region with an unknown boundary is transformed into a problem for the half-space. The plane-parallel flow (1), (7) is most simply investigated by using functions of a complex variable.

In the multiply connected region D^- of the complex plane $z=x+iy$ we will introduce the following analytic function (the complex potential):

$$W(z,t)=\phi(x,y,t)+i\psi(x,y,t), \quad z\in D^-, \tag{9}$$

where $\phi(x,y,t)$ is the hydrodynamic potential, $\psi(x,y,t)$ is the stream function, and the variable t is a parameter. We will use the complex potential only for those values of the parameter t for which the Cauchy–Riemann conditions hold, i.e., $\phi_x=\psi_y$, $\phi_y=-\psi_x$. Accordingly, we will define the analytic functions $V(z,t)$ (complex velocity) and $S(z,t)$ (the Keldysh function)[21] in the region D^- by the relations

$$V(z,t)=\bar{W}'(z,t)=v_1(x,y,t)+iv_2(x,y,t), \tag{10}$$

$$S(z,t)=iW_{tt}(z,t)-W'(z,t), \quad z\in D^-. \tag{11}$$

Here v_1 and v_2 are projections of the hydrodynamic velocity vector $\mathbf{v}$ onto the x and y axis: $v_1(x,y)=\mathrm{Re}W'$, $v_2(x,y)=-\mathrm{Im}W'$. The bar over the functions signifies complex conjugation, and the dash, differentiation with respect to the complex variable z.

The limiting values of these functions as we approach the real axis from below ($y\to-0$) are denoted by the same lower-case letters

$$\lim_{y\to-0} W(z,t)\to w(x,t), \qquad \lim_{y\to-0} \bar{V}(z,t)\to\bar{v}(x,t)=w_x,$$

$$\lim_{y\to-0} S(z,t)\to s(x,t)=iw_{tt}-w_x.$$

In this notation, boundary condition (7) for the limiting value of the complex potential $w(z,t)$ leads to the expression

$$w^{(1)}(x,t)+w^{(2)}(x,t)+w^{(3)}(x,t)+\ldots=0, \tag{12}$$

where

$$w^{(1)}=\mathrm{Im}(iw_{tt}-w_x), \qquad w^{(2)}=(|w_x|^2)_t-\mathrm{Re}w_t\cdot\mathrm{Res}_x,$$

$$w^{(3)}=\mathrm{Re}(w_x^2\bar{w}_{xx})-\mathrm{Re}w_t\{2\mathrm{Im}(\bar{w}_{xx}w_x)_t+\mathrm{Im}\bar{w}_{xt}\cdot\mathrm{Res}_x$$
$$+\tfrac{1}{2}\mathrm{Re}w_t\cdot\mathrm{Im}s_{xx}\}-\tfrac{1}{2}|w_x|^2\cdot\mathrm{Res}_x .$$

Here and in what follows we denote the real and imaginary part of the complex function w by $\mathrm{Re}w$, $\mathrm{Im}\, w$. A distinctive feature of the boundary condition (12) in the complex-potential representation is the fact that it contains not only the limiting function $w(x,t)$ but also the complex conjugate function $\overline{w(x,t)}$. In order to obtain an equation for the complex potential $W(z,t)$ over the entire region $z\in D^-$ we use methods that involve the linear adjoint boundary value problem,[22,23] for which D^- means the whole lower half-plane except for isolated singular points of the flow. Accordingly, the region D^+ is the mirror reflection of the region D^- with respect to the real axis. We use the functions $W(z,t)$, $S(z,t)$, $z\in D^-$ to construct the functions $W_*(z,t)$ and $S_*(z,t)$ in the region D^+ of the upper half-plane by invoking the reflection symmetry implied by the Schwartz principle:

$$W_*(z,t)=\overline{W(\bar{z},t)}, \quad S_*(z,t)=\overline{S(\bar{z},t)}, \quad z\in D^+.$$

The limiting values of these functions as $y\to+0$ are $\overline{w(x,t)}$ and $\overline{s(x,t)}$, respectively. By introducing the function $G(z,t)$, which is piecewise analytic over the entire complex plane,

$$G(z,t)=\begin{cases} S_*, & z\in D^+ \\ S, & z\in D^-. \end{cases}$$

we can write the boundary condition (12) in terms of the discontinuity in the function $G(z,t)$ as we pass through the real axis:

$$G^+-G^-=2i[w^{(2)}(x,t)+w^{(3)}(x,t)+\ldots]. \tag{13}$$

In the linear theory there is no discontinuity; consequently, the function $S_*(z,t)$ is the direct analytic continuation of the function $S(z,t)$ through the real axis, and $G(z,t)$ is analytic in the entire finite part of the complex plane. In our case it is assumed that the complex potential $W(z,t)$, and consequently the function $S(z,t)$ as well, have isolated poles in the region occupied by the fluid. Let us write these functions in the form of a sum of two parts: a part that is regular in the lower half-space, and the principal part of a Laurent expansion in the vicinity of all the poles. Thus, $W(z,t)=W_R+W_P$, $S(z,t)=S_R+S_P$. Then the discontinuity and poles can be used to reconstruct the regular function $S_R(z,t)$ via the Cauchy integral:

$$S_R(z,t)+\overline{S_P(\bar{z},t)}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{w_2(\xi,t)+w_3(\xi+t)+\ldots}{\xi-z}d\xi-A_0,$$
$$z\in D^-. \tag{14}$$

Here we assume that the flow velocity at infinity can have a constant component, i.e., the point at infinity is a simple pole of the function $W(z,t)$. Accordingly, the limiting complex function $w(x,t)=w_R+w_P$ for the homogeneous boundary value problem (12) must be found by solving a nonlinear integrodifferential equation with dimensions (1+1):

$$(w_R+\overline{w_P})_{tt}+i(w_R-\overline{w_P})_x$$
$$+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{w_2(\xi,t)+w_3(\xi,t)+\ldots}{\xi-x+i0}d\xi=iA_0. \tag{15}$$

With this nonlinear equation as a basis, we can find all admissible distributions $w_R(x,t)$ and $w_P(x,t)$ within the weakly nonlinear approximation that describe intrinsic types of plane-parallel flow containing isolated singularities for an ideal infinitely deep gravitating fluid with a free surface. For this homogeneous equation, the Cauchy initial value problem can be formulated exactly as it is in the usual linear theory of the motion of a body under the surface of a fluid, where the sources $W_P(z,t)$ are assumed to be prespecified.[1]

3. STEADY-STATE MULTIPOLE SOLUTIONS TO THE NONLINEAR EQUATION IN THE QUADRATIC APPROXIMATION

Let us discuss some special steady-state solutions to the basic nonlinear Eq. (15) taken from the class of rational functions; for this we include only the quadratic nonlinear terms in the expression under the integral sign. Let us introduce a real spectral parameter λ, namely the dimensionless velocity of a steady-state perturbation propagating in the positive direction of the x axis. We then introduce a system of coordinates moving with this perturbation $\theta=x-\lambda t$, and look for a characteristic function of the form $w_\theta=u(\theta)$. This procedure leads to the expression

$$\lambda^2(u_R+\overline{u_P})_\theta+i(u_R-\overline{u_P})$$

$$+\frac{i\lambda}{\pi}\int_{-\infty}^{\infty}\frac{-(|u|^2)_\xi+\mathrm{Re}u\cdot\mathrm{Re}(i\lambda^2u_{\xi\xi}-u\xi)}{\xi-\theta+i0}d\xi=iA_0. \tag{16}$$

Let us further assume that the function $w'_R(z)$ has poles in the upper half-plane. Then the general form of the complete function $w'(z)$ can be written as a multipole expansion, with poles located in an arbitrary way over the entire finite part of the complex plane. In this section we will look for rational solutions to Eq. (16) of a more special form, by postulating that the function $w'(z)$ has M paris of poles symmetrically located relative to the real axis. As we will show below, this ansatz to (16) is the only one whose parameters are determined uniquely by a closed system of algebraic equations of finite order. The coefficients of the multipolar expansions in the neighborhood of all pairs, and also the expansion coefficients for each individual pair, are assumed to be constant and unequal to each other. For the limiting function $u(\theta)$ we will seek a solution in the form

$$u^{(M)}_{\{N_m\}}(\theta)=u_R(\theta)+u_P(\theta),$$

$$u_R(\theta)=A_0+\sum_{m=1}^{M}\sum_{n=1}^{N_m}\frac{A_{nm}}{(\theta-z_m)^n},$$

$$u_P(\theta)=\sum_{m=1}^{M}\sum_{n=1}^{N_m}\frac{\bar{B}_{nm}}{(\theta-\bar{z}_m)^n}, \tag{17}$$

where $z_m=x_m+iy_m$, $y_m>0$; here A_0 is real, and A_{nm}, B_{nm} are complex constants. M is the number of pairs of poles, $\{N_m\}=N_1, N_2, ..., N_m$ is the multiplicity distribution of the paired poles of the function $V(z)$. For fixed M and $\{N_m\}$ the number of arbitrary parameters to be determined equals $2(M+1+\Sigma_{m=1}^{M}N_m)$. It is obvious that if the complex function $u(\theta)$ is either exclusively real or exclusively imaginary, the coefficients of the expansion (17) satisfy the relations $B_{nm}=A_{nm}$ or $B_{nm}=-A_{nm}$, respectively. The complex flow potential $W(z)$ in the region D^- is determined by the expressions

$$W(z)=A_0z+\sum_{m=1}^{M}\left\{W_{0m}(z)+\sum_{n=1}^{N_m-1}W_{nm}(z)\right\}, \tag{18}$$

where

$$W_{0m}(z)=A_{1m}\ln(z-z_m)+\bar{B}_{1m}\ln(z-\bar{z}_m),$$

$$W_{nm}(z)=-\frac{1}{n}\left(\frac{A_{n+1,m}}{(z-z_m)^n}+\frac{\bar{B}_{n+1,m}}{(z-\bar{z}_m)^n}\right).$$

In substituting (17) into the original expression (16), we have made important use of the fact that expanding the function to be integrated in simple fractions allows us to write it in the form of a sum of two functions, one of which is analytic in the lower half-plane and the other in the upper half-plane. In calculating the products of functions of the form (17) we make use of the following expression for decomposing a product into simple fractions:

$$\frac{1}{(z-z_m)^n(z-\bar{z}_{m'})^{n'}}=\sum_{k=1}^{n}\frac{G^{mm'}_{n-k,n'}}{(z-z_m)^k}+\sum_{k=1}^{n'}\frac{\bar{G}^{m'm}_{n'-k,n}}{(z-\bar{z}_{m'})^k}, \tag{19}$$

$$\frac{1}{(z-z_m)^n(z-z_{m'})^{n'}}=\sum_{k=1}^{n}\frac{Q^{mm'}_{n-k,n'}}{(z-z_m)^k}+\sum_{k=1}^{n'}\frac{Q^{m'm}_{n'-k,n}}{(z-z_{m'})^k},$$

$$m\neq m', \tag{20}$$

where

$$G^{mm'}_{nn'}=\frac{(-1)^n\binom{n+n'-1}{n}}{(z_m-\bar{z}_{m'})^{n+n'}},\quad Q^{mm'}_{nn'}=\frac{(-1)^n\binom{n+n'-1}{n}}{(z_m-z_{m'})^{n+n'}},$$

$$Q^{mm}_{nn'}\equiv 0. \tag{21}$$

For the expansion coefficients G, Q, we will require in what follows that the following symmetry relations hold

$$G^{m'm}_{n'n}=(-1)^{n+n'}\frac{n}{n'}G^{mm'}_{nn'},\quad G^{mm'}_{nn'}=(-1)^{n+n'}\overline{G}^{m'm}_{nn'},$$

$$G^{mm'}_{k-l,n+l}=(-1)^l\frac{k(k-1)\ldots(k-l+1)}{n(n+1)\ldots(n+l-1)}G^{mm'}_{kn}. \tag{22}$$

These relations also hold for Q.

Using the rule for interchanging the order of summation,

$$\sum_{n=1}^{N}\sum_{k=1}^{n}a_{nk}=\sum_{k=1}^{N}\sum_{n=k}^{N}a_{nk}, \tag{23}$$

we obtain a system of nonlinear algebraic equations for each pair of poles $m=1, ..., M$ that determines the free parameters A_{nm}, B_{nm}, z_m, and λ:

$$A^-_{1m}-i\lambda\sum_{m'=1}^{M}V^{mm'}_1=0,\quad n=1, \tag{24}$$

$$L_{nm}+\sum_{k=1}^{n-2}[2(n-1)A_{km}B_{n-k-1,m}+A^+_{km}F_{n-k-2,m}]$$

$$+\sum_{m'=1}^{M}[(n-1)U^{mm'}_n+V^{mm'}_n]=0,\quad n=\overline{2,N_m+2}, \tag{25}$$

$$\sum_{k=n-N_m-2}^{N_m}[2(n-1)A_{k+1,m}B_{n-k-2,m}+F_{km}A^+_{n-k-2,m}]=0,$$

$$n=\overline{N_m+3,2N_m+1}, \tag{26}$$

$$A^+_{N_mm}A^-_{N_mm}=0,\quad n=2N_m+2, \tag{27}$$

where we have introduced the notation

$$A^\pm_{nm}=A_{nm}\pm B_{nm},\quad A^\pm_{nm}\neq 0,\quad \text{if } n=\overline{1,N_m},$$

$$F_{nm}=\frac{N+1}{2}(A^+_{n+1,m}+i\lambda^2nA^-_{nm}),\quad F_{nm}\neq 0,\quad \text{if } n=\overline{0,N_m},$$

$$L_{nm}=\frac{i}{\lambda}A^-_{nm}-(n-1)(\lambda-2A_0)A^+_{n-1,m}+2A_0F_{n-2,m},$$

$$U^{mm'}_n=\sum_{n'=1}^{N_{m'}}\sum_{k=n}^{N_{m+1}}[G^{mm'}_{k-n,n'}(A^+_{k-1,m}\overline{A^+_{n'm'}}+A^-_{k-1,m}\overline{A^-_{n'm'}})$$

$$+Q^{mm'}_{k-n,n'}(A^+_{k-1,m}A^+_{n'm'}-A^-_{k-1,m}A^-_{n'm'})],$$

$$V^{mm'}_n=\sum_{n'=1}^{N_{m'}+2}\sum_{k=n}^{N_m+2}[G^{mm'}_{k-n,n'}(A^+_{km}\overline{F_{n'-2,m'}}+F_{k-2,m}\overline{A^+_{n'm'}}$$

$$+Q^{mm'}_{k-n,n'}(A^+_{km}F_{n'-2,m'}+F_{k-2,m}A^+_{n'm'})]. \tag{28}$$

The total number of complex equations in the system (24)–(27) equals $2(M+\Sigma_{m=1}^{M}N_m)$, while the number of free parameters of the solution (17) is two more than this. By virtue of the translation invariance of the equations, we can always put one pair of poles, for example, the first, on the y axis, i.e., $x_1=0$, and choose to normalize the ordinates of the poles by the length l, so that $y_1=\pm1$. Once this is done, we have enough algebraic equations to unambiguously determine the real parameter λ, the multiple moments, and the coordinates of the remaining pairs of poles, for any M and $\{N_m\}$. Numerical analysis of the algebraic system (24)–(27) shows that it has nontrivial solutions for any M and N_m with the exception of several of the simplest cases. In particular, we can show by direct substitution that steady-state solutions of the form (17) do not exist for any number of pairs of simple poles ($N_m=1$, $m=\overline{1,M}$) for any set of relative positions of these poles. Likewise, the case of a single pair of poles ($M=1$) has nontrivial solutions only for $N_1\geqslant6$. One such solution ($N_1=6$) will be given below. It is obvious that this situation is associated exclusively with the quadratic nature of the weakly nonlinear Eq. (16) if we note that either linearization or inclusion of cubic nonlinear terms always lead to an infinite system of coupled algebraic equations for the coefficients of expansion (17).

Relative to the coordinate system moving with the wave, the flow is steady-state and can be described by a complex potential $\tilde{W}=-\lambda z+W(z)$. In this case, the velocity vector at each point does not change with time, and consequently the trajectories of fluid particles coincide with the flux lines of the flow:[11]

$$\tilde{\psi}(x,y)=\mathrm{Im}\tilde{W}(z)=-\lambda y+\mathrm{Im}W(z)=\mathrm{const}. \tag{29}$$

The zero flux line corresponds to a velocity that equals A_0 at infinity.

Note also that within the framework of the quadratic approximation, the second half of the algebraic system [Eqs. (26) and (27)] describes only the interaction of the multipoles within a pair. In this case the last M equations of (27) show that the moments of the highest multipoles in each pair must satisfy one of the relations $A_{N_m,m}=\pm B_{N_m,m}$. Physically this implies that the velocity fields created by these poles have either only horizontal components ($A^-_{N_m,m}=0$) or only vertical components $A^+_{N_m,m}=0$) on the real axis. In both cases, Eqs. (26) allow us to compute all the remaining moments A^-_{nm} (or A^+_{nm}) of lower order in terms of A^+_{nm} (or A^-_{nm}) recursively. Let us introduce expressions for the last three nonzero coefficients (omitting the label m):

a) $A^+_N\neq0,\quad A^-_N=0$:

$$A^-_{N-1}=\frac{3i}{\lambda^2(N-1)}A^+_N,\quad A^-_{N-2}=\frac{3i}{\lambda^2(N-2)}A^+_{N-1},$$

$$A^-_{N-3}=\frac{3i}{2\lambda^2(N-2)(N-3)}\left[A^+_{N-2}+\frac{6}{\lambda^4(N-1)}A^+_N\right], \tag{30}$$

b) $A^-_N\neq0,\quad A^+_N=0$:

$$A^+_{N-1}=-\frac{2i}{\lambda^2(N+1)}A^-_N,$$

$$A^+_{N-2}=-\frac{2i(N^2+2N-1)}{\lambda^2N(N+1)^2}A^-_{N-1},$$

$$A^+_{N-3}=-\frac{2i(N-1)}{\lambda^2N(N+1)^3}\left[\frac{6}{\lambda^4}A^-_N+2\frac{(A^-_{N-1})^2}{A^-_N}+(N+1)(N+4)A^-_{N-2}\right].$$

By doing this we reduce the total number of equations for the algebraic system by half.

An important simplification in finding solutions to the complex system (24)–(27) is achieved if we assume that all the poles are located on the vertical axis: $z_m=iy_m$. In this case we can show that new unknowns can be introduced, defined by the relations

$$\begin{pmatrix}a_{nm}\\ b_{nm}\end{pmatrix}=(-1)^{(n+1)/2}i^n\begin{pmatrix}A_{nm}\\ B_{nm}\end{pmatrix}, \tag{31}$$

in terms of which the system becomes real.

Truncated nonlinear equation

An important simplification of the basic integral equation of the quadratic theory (16) results when we undertake to find solutions that are "close" to solutions of the linearized equation, i.e.,

$$|\lambda^2u_\theta+iu|\ll|u|. \tag{32}$$

Direct substitution of any finite expansion of the form (17) into the linearized equations does not lead to a closed system of algebraic equations. However, if we neglect the second term in the numerator of expression (16) under the integral sign by virtue of inequality (32), we obtain a truncated model equation for the quadratic theory for sufficiently smooth solutions close to the solution of the linearized equation:

$$\lambda^2(u_R+\overline{u_P})_\theta+i(u_R-\overline{u_P})-\frac{i\lambda}{\pi}\int_{-\infty}^{\infty}\frac{(|u|^2)\xi}{\xi-\theta+i0}d\xi=iA_0. \tag{33}$$

By truncating Eqs. (16), we obtain a closed system for the expansion coefficients of the form (17), consisting of a finite number of algebraic equations, even if the poles that lie in the upper half-plane of the function $u_R(z)$, which is regular in D^-, are not the complex conjugates of the poles of the function $u_P(z)$ in the lower half-plane.

Let us introduce the complex potential $W(z)$ with M poles of multiplicity N_m-1 located arbitrarily in the finite part of the complex z plane (and not too close to the real axis). For simplicity we will assume that the fluid is at rest at infinity ($A_0=0$). Let M_1 terms of the expansion with Im $z_m > 0$ form the regular part of the potential W_R in the lower half-plane, while the remaining $M-M_1$ terms with Im $z_m<0$ form the singular part W_P. Because the integers M_1 and $M>M_1$ are given arbitrarily, no special decomposition of the function $W(z)$ into two terms is necessary:

$$W(z)^{(M)}_{\{N_m\}}=W_R(z)+W_P(z)=\sum_{m=1}^{M}\sum_{n=1}^{N_m-1}\frac{C_{nm}}{(z-z_m)^n},$$

$$W'(z)=\sum_{m=1}^{M}\sum_{n=2}^{N_m}\frac{A_{nm}}{(z-z_m)^n}, \qquad (34)$$

where $A_{nm}=-(n-1)C_{n-1,m}$. The upper and lower labels on the potential function denote the total number of poles and their multiplicity distributions, respectively. Substituting (34) into the truncated Eq. (33) leads to the following system of nonlinear algebraic equations for the unknowns C_{nm}, z_m, λ (Ref. 24):

$$iC_{nm}-2\lambda\sum_{m'=1}^{M}\sum_{k'=1}^{N_{m'}-1}\sum_{k=1}^{N_m-n}\frac{(-1)^k\binom{k+k'}{k}k'(k+n-1)}{(z_m-\bar{z}_{m'})^{k+k'+1}}$$

$$\times C_{k+n-1,m}\bar{C}_{k'm'}=0,$$

$$m=\overline{1,M};\quad n=\overline{1,N_m-1}, \qquad (35)$$

$$\lambda=-2\sum_{m'=1}^{M}\sum_{k'=1}^{N_{m'}-1}\frac{k'\bar{C}_{k'm'}}{(z_m-\bar{z}_{m'})^{k'+1}},\quad m=\overline{1,M}\ . \qquad (36)$$

As in the case of pairwise conjugate poles, we may assume that $x_1=0$, $y_1=1$, and all the remaining unknown parameters are determined from the system $\Sigma_{m=1}^{M} N_m$ complex Eqs. (35) and (36). Among the set of solutions with real λ, only those that satisfy the criteria (5) and (32) will be physically meaningful over the entire real axis.

A formally nonlinear algebraic system (35), (36) can be obtained from (24)–(27) by setting $A_0=A_{1m}=B=F=0$ there.

In the simplest case $N_m=2$, $m=\overline{1,M}$, i.e., when the complex potential consists of a flow created by a system of dipoles, the algebraic system (35) becomes linear with respect to the moments C_{1m}:

$$\sum_{m'=1}^{M}\frac{\bar{C}_{1m'}}{(z_m-\bar{z}_{m'})^2}+\frac{\lambda}{2}=0,$$

$$\sum_{m'=1}^{M}\frac{\bar{C}_{1m'}}{(z_m-\bar{z}_{m'})^3}+\frac{i}{4\lambda}=0,\quad m=\overline{1,M}. \qquad (37)$$

This allows us to reduce the number of equations of the nonlinear system (35), (36) to M for the unknowns z_2, ..., z_M, λ.

Thus, the primary result of this general analysis is that in a weakly nonlinear approximation where only quadratic nonlinear terms are included the basic Eq. (16) and its truncated variant (33) admit the existence of the rational solutions (17) and (34) respectively which have the form of steady-state vorticity-bearing gravitational waves whose flows have a complex multipolar structure near the free surface of the ideal gravitating fluid. In the next section we will find a number of exact solutions to Eqs. (16) and (33) that satisfy the criteria (5), (32) of the theory.

4. EXACT SOLUTIONS OF THE TRUNCATED EQUATION

4.1. Series of single-pole solutions ($M=1$)

Without loss of generality let us place the single pole on the imaginary axis and write the expression for the complex potential according to (34) in the form

$$W_N^{(1)}(z)=\sum_{n=1}^{N-1}\frac{C_n}{(z-iy_1)^n}. \qquad (38)$$

Here $C_n=C_n$, $N_1=N$ is the multiplicity of the pole of the function $W'(z)$, and $z=x-\lambda t+iy$, as in the previous section. If we introduce a normalization of the amplitudes into (35)

$$C_n=(-2iy_1)^{(n+1)}\frac{y_1}{\lambda n}Y_n,$$

then the real spectral parameter λ is given in explicit form:

$$\sum_{k=1}^{N-n}\sum_{k'=1}^{N-1}\binom{k+k'}{k}Y_{k+n-1}\bar{Y}_{k'}-\frac{1}{n}Y_n=0,\quad n=\overline{1,N-1}, \qquad (39)$$

$$\lambda^2=-2y_1\sum_{k=1}^{N}\bar{Y}_k. \qquad (40)$$

It is clear, at any rate, that any real solutions to system (39) give a potential $W_N^{(1)}(z)$ that is a characteristic function of the spectral problem (33). In this case the requirement that λ take on real values is ensured by an appropriate choice of sign for y_1 in (40), i.e., locating the pole outside the fluid ($y_1>0$) or inside it ($y_1<0$). The structure of the algebraic system (39) itself reveals that the nonlinear terms describe only the interaction between multipoles and their "images" relative to the plane $y=0$. There is no direct interaction of the multipoles with one another in the truncated equations. In nonlinear Fourier analysis, this approximation corresponds to taking into account only the effects of detection and discarding higher harmonics. Let us consider the first few exact solutions to the system (39).

In the simplest dipolar case ($N=2$)

$$Y_1=\frac{1}{2},\quad \lambda^2=-y_1,\quad C_1=-\frac{2y_1^3}{\lambda}, \qquad (41)$$

a solution exists if the pole is located in the fluid itself ($y_1<0$). Choosing as the characteristic length the distance from the pole to the unperturbed surface, i.e., $l=|y_1|=1$, we obtain the following expressions for the potential $\tilde{W}(z)$ in the system of coordinates moving with the wave

$$\tilde{W}(z) = -z + \frac{2}{z+i}.$$

This potential is a superposition of two flows: a uniform flow and a dipole whose axis is oriented along the flow. There are two critical points for the flow $[\tilde{W}'=0$: $y_{\pm}=i(-1\pm\sqrt{2})]$, located symmetrically with respect to the dipole center. Equation (29) for the flux lines has the form

$$(y+y_0)[x^2+(y+1)^2]+2(y+1)=0,$$

where y_0 is an arbitrary real constant. The flux lines that correspond to the values $1-2\sqrt{2}<y_0<1+2\sqrt{2}$ are focused by the dipole at its center. Consequently, the zero flux line, which is the profile of the free surface, also passes through the center of the dipole and has the form of a trough of unit depth. The solution (41) does not satisfy the criterion for weak nonlinear flow (5) near the crest ($x=0$), although for $|x| \to \infty$ the velocity field decays like $|x|^{-2}$; accordingly, the profile of the solitary wave decays in the same way. In a certain sense this solution is similar to the singular soliton solution of the exact Crapper equation for pure capillary waves.[9,10] However, in our case the approximations we have used limit the applicability of our solution to regions far from the crest.

Analysis of the algebraic system (39) shows that solutions exist only for $y_1<0$, and for $N=2$, 3, 4 there are only real roots whose number increases rapidly. Thus, for $N=3$ there are two of these roots $Y_{1,2}^{(j)}$, $j = 1,2$:

1) $j=1$,

$$\begin{gathered} Y_1^{(1)}=2-\sqrt{3}, \quad C_1^{(1)}=2|y_1|^{5/2}(2-\sqrt{3}), \\ Y_2^{(1)}=-\frac{1}{2}+\frac{1}{\sqrt{3}}, \quad C_2^{(1)}=2i|y_1|^{7/2}\left(\frac{2}{\sqrt{3}}-1\right), \\ (\lambda^2)^{(1)}=|y_1|\left(1-\frac{1}{\sqrt{3}}\right), \end{gathered} \tag{42}$$

2) $j=2$,

$$\begin{gathered} Y_1^{(2)}=2+\sqrt{3}, \quad C_1^{(2)}=2|y_1|^{5/2}(2+\sqrt{3}), \\ Y_2^{(2)}=-\frac{1}{2}-\frac{1}{\sqrt{3}}, \quad C_2^{(2)}=-2i|y_1|^{7/2}\left(\frac{2}{\sqrt{3}}+1\right), \\ (\lambda^2)^{(2)}=|y_1|\left(1+\frac{1}{\sqrt{3}}\right). \end{gathered} \tag{43}$$

It is clear that the velocity and amplitude of the crest in the first solution ($j=1$) is smaller than for the case $N=2$; however, the flux lines of the flow caused by the system dipole plus quadrupole, along with the corresponding free surface, are "sucked into" the point $y=-y_1$ as before.

For $N=4$ the number of real solutions is now four; one of them ($j=1$) yields values of the velocity and amplitude smaller than for $N=2$, 3. This tendency persists as N increases, resulting in an entire branch of solutions $W_N^{(1)}(z)$. At the same time, as N increases, new branches form, corresponding to the values $j=2$, 3, ... For $N=5$ the number of real roots of the system (39) equals six, for $N=6$ it equals ten, for $N=7$ it equals 14, etc. We have calculated the branch corresponding to label $j=1$ numerically up to $N=19$. We were unable to make a complete classification of the solutions to Eq. (39) for arbitrary N, as can be done, e.g., for the states of an electron in the hydrogen atom, although corresponding changes in notation allow Eq. (15) itself to be written as a one-dimensional Schrödinger equation for a free electron with self-interaction.

Since no general methods exist for solving systems of nonlinear algebraic equations, we used the Newton–Raphson method to find special solutions for $N>4$, and also for cases where $M>1$. In connection with this, it is worth noting that almost all the solutions were obtained numerically, and yet we use the term exact solution. In this way we underline a specific feature of this type of problem: when a finite series is substituted into an integral equation, the latter reduces to a closed system of a finite number of nonlinear algebraic equations. In this case, the accuracy of the solution is determined only by the precision of the computer used. Usually, however, the Galerkin procedure leads to an infinite system of algebraic equations and, after replacing it by a finite system of N equations, it is necessary to prove convergence of the solution as $N\to\infty$. In our case, this situation obtains whether we solve the linearized Eq. (33) using the expansion (34) or take into account the cubic nonlinear terms in this equation.

4.2. Series of two-pole solutions (*M*=2)

Let us consider a case where sources z_m are located on the single vertical line $x=0$: $z_m=iy_m$. According to (31), the moments of the complex potential (34) have the form

$$C_{nm}=(-1)^{(n+1)/2}i^{n+1}c_{nm}, \tag{44}$$

where c_{mn} are real numbers. The complex system of algebraic equations (35), (36) reduces to a system of real equations:

$$c_{nm}-2\lambda\sum_{m'=1}^{M}\sum_{k'=1}^{N_{m'}-1}\sum_{k=1}^{N_m-n}\frac{\beta_{nkk'}\binom{k+k'}{k}k'(k+n-1)}{(y_m+y_{m'})^{k+k'+1}} \times c_{k+n-1,m}c_{k'm'}=0, \quad m=\overline{1,M}; \quad n=\overline{1,N_m-1}, \tag{45}$$

$$\lambda=2\sum_{m'=1}^{M}\sum_{k'=1}^{N_{m'}-1}\frac{(-1)^{k'/2}k'c_{k'm'}}{(y_m+y_{m'})^{k'+1}},$$

$$\beta_{nkk'}=(-1)^{(n/2)+(k'/2)+(n+k+1)/2}. \tag{46}$$

We will assume that poles z_1 and z_2 are located on different sides of the unperturbed surface $y=0$. As N_1 and N_2 increase, the number of solutions grows rapidly and we were unable to make a complete classification of them. However, we were successful in identifying several series of solutions which, for different N_1 and N_2, correspond to a common spatial configuration of the characteristic modes, e.g., a symmetric one-humped solitary wave, a two-humped wave, etc. Naturally, in this case, the branch of the solutions that is most interesting is the one that begins with certain values $N_{1,2}$ that satisfy the criteria (5), (32). Here we present results

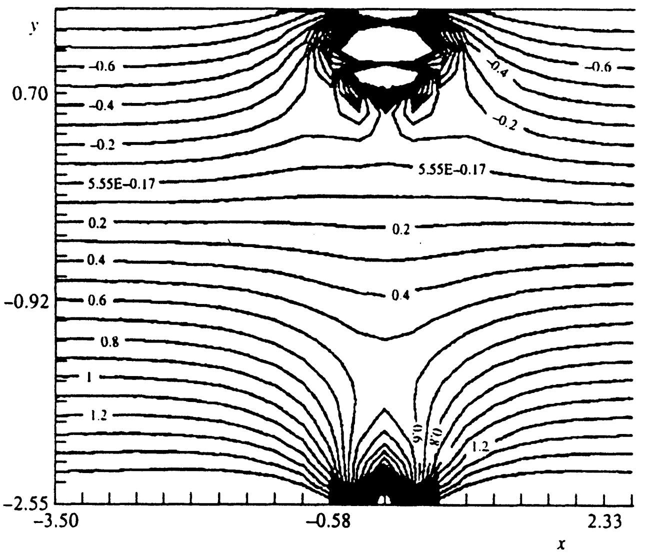

FIG. 1. Flux lines for a flow with complex potential $\tilde{W}^{(2)}_{4,3}(z)$.

for only one such solution branch of the system (45), (46), for which the pole z_1 with multiplicity $N_1=N+1$ is located above the fluid, while the pole z_2 is located in the fluid and has multiplicity $N_2=N$. If we choose the height of the pole z_1 above the unperturbed surface as the characteristic length l, i.e., $y_1=1$, then we should seek the complex potential $W^{(2)}_{N+1,N}(z)$ in the form

$$W^{(2)}_{N+1,N}(z)=\sum_{n=1}^{N+1}\frac{\alpha_n c_{n,1}}{(z-i)^n}+\sum_{n=1}^{N}\frac{\alpha_n c_{n,2}}{(z-iy_2)^n},\quad z\in D^-, \tag{47}$$

where $\alpha_{2n}=i$, $\alpha_{2n+1}=1$; the moments $c_{n,1}$, $c_{n,2}$ are defined in (44).

In our paper Ref. 24 listed all the parameters used in Eq. (47) for a series of seven solutions $N=\overline{2,8}$, all of which describe a solitary wave. It turns out that as the number of terms in the expansion in (47) increases, all of its parameters, i.e., c_{nm}, λ, $\eta(0)$ decrease monotonically. The values of the moments $c_{n,1}$, $c_{n,2}$ for the same n and N approach each other as N increases, while the amplitudes of the tail in the expansion (47) rapidly reduce to zero. A specific feature of the Newton–Raphson method is that for a set of solutions that lie on the same branch, the radius of convergence of the process that iterates toward them decreases as N increases. Therefore, we used a special approximation program based on the previous $N-1$ points to choose trial values of the unknowns for successive values of N.

Figure 1 shows a portrait of the flux lines for the flow with potential $W^{(2)}_{4,3}(z)$ in accordance with the definition (29). The numerical parameters of the flow are given in Table I. The zero flux line corresponds to the profile of the free surface. It is clear that in the immediate vicinity of this surface, where only the nonlinear boundary condition (3) need be fulfilled, the velocity field is quite smooth and the criterion for the approximate theory can be satisfied with arbitrary accuracy for sufficiently large N.

TABLE I.

n	c_{n_1}	c_{n_2}
1	0.318561	0.324607
2	0.229794	0.203595
3	−0.102268	−0.061135
4	−0.021191	–
	$y_2=-2.7613$	$\lambda=0.473840$

In order to reproduce the profile of the free surface $\eta\phi$ using Eq. (8) based on the potential given by the quadratic approximation, it is sufficient to use only the linear relation between η and ϕ, since we used only this linear relation in calculating the quadratic terms in (7). Because the free surface is by definition a flux line, an alternative method can be used to calculate the profile $\eta(\theta)$ as the zero flux line (29); this method entirely confirms the conclusion arrived at above. In addition, this also implies that the approximate small-amplitude solutions are themselves correct with respect to the exact Eq. (3). In Fig. 2 we show the results of computing the function $\eta(\theta)$ for even values of $N=2, 4, 6, 8$. The first two solutions $N=2, 3$ exhibit rather small troughs at the top of the hump, which disappear for subsequent N. For $N=8$, in dimensional variables with $l=100$ m we obtain for the height of the solitary wave crest a value $h^{(8)}=l\eta^{(8)}\approx 1$ m, along with the rather high velocity $c^{(8)}=\lambda^{(8)}\sqrt{gl}\approx 7.4$ m/s.

4.3. Series of multipole solutions

Assume we have $2M+1$ sources (poles) located on a single vertical line and creating a flow in the region occupied by a fluid with a potential of the form (34). The flow structure, i.e., the relative position and labeling of these poles, is in this case determined by the form of the specific expansion

$$W^{(2M+1)}_{\{N_m\}}(z)=\sum_{m=-M}^{M}\frac{c_{1,m}}{z-iy_m}+i\frac{c_{2,M}}{(z-i)^2},$$

$$\{N_m\}=\{1,1,\ldots,1;2\}. \tag{48}$$

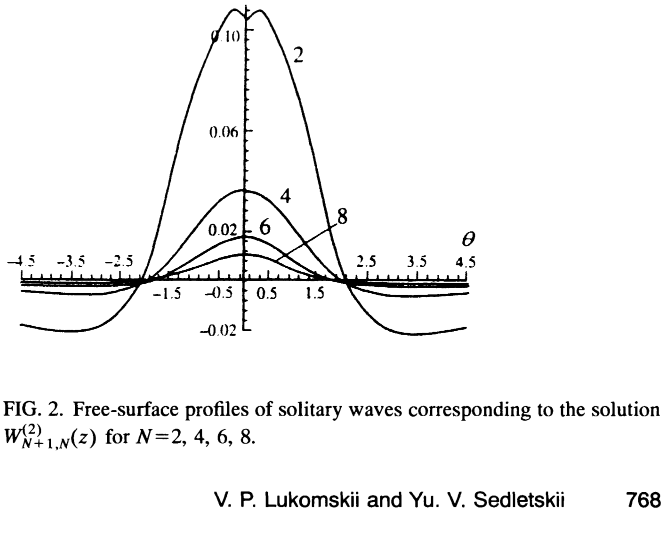

FIG. 2. Free-surface profiles of solitary waves corresponding to the solution $W^{(2)}_{N+1,N}(z)$ for $N=2, 4, 6, 8$.

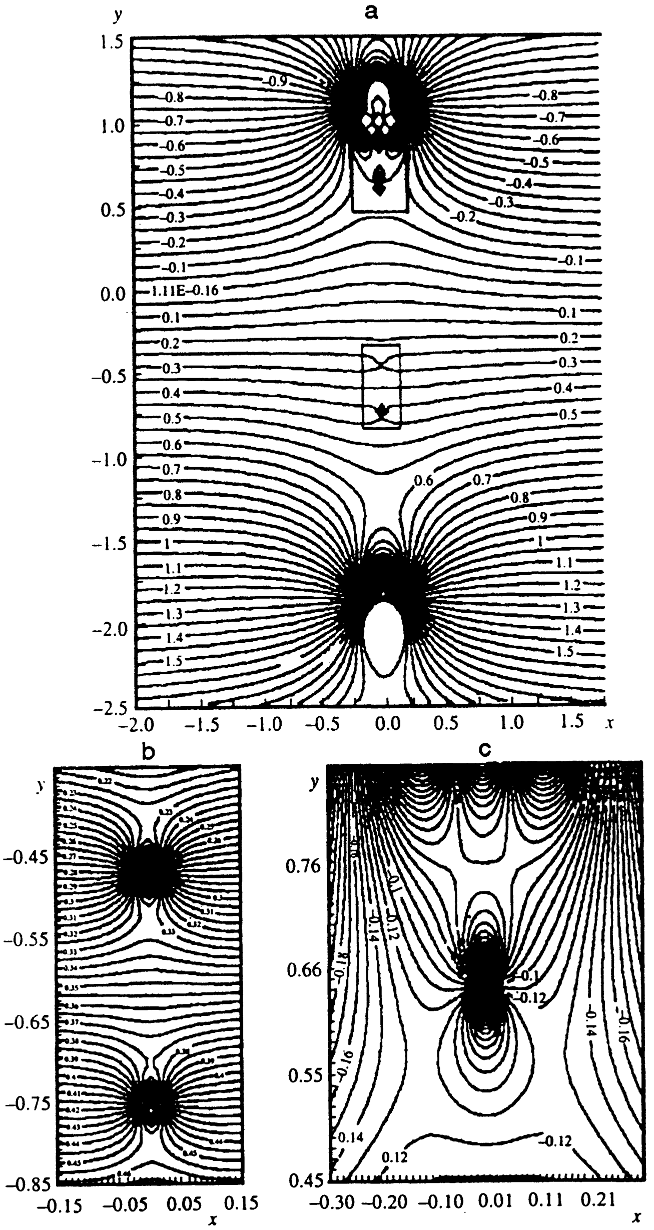

FIG. 3. Flux lines of a flow with complex potential $\tilde{W}^{(5)}_{\{1,1,1,1;2\}}(z)$.

Here the labeling is such that the larger the label of the pole, the higher on the y axis it is located. Furthermore, for all values of M, the poles with negative labels $m=\overline{-M,0}$ are located in the fluid, while those with positive $m=\overline{1,M}$ are located outside it. The source with label $m=M$ (above the fluid) has a dipole+quadrupole structure, while the remaining poles are dipoles. As the characteristic length l we choose the height of the pole M above the unperturbed surface: $y_M=1$. In this configuration, we studied one branch of the solutions to (48) up to $M=17$ inclusive. This branch corresponded to symmetric solitary waves with a single crest. The main sources that give the largest contribution to the flow are the edge solutions with labels $m=M$ and $m=-M$. Each new solution was obtained by adding a pair of poles between the main ones. As in the previous case, during the iterations, trial values for each new solution were found based on the previous one, using a special approximation program. Computations were carried out using Newton's method with double precision. For a given value of M, the corresponding system of nonlinear algebraic equations was of order $4M+3$. The parameters of the exact solutions (48) for $M=\overline{1,6}$ and $M=17$ are given in our Ref. 24. It turns out that as M increases the lowest pole y_{-M} approaches the value $y=-1$ from outside. The remaining poles $1\leqslant m\leqslant M-1$ and $-M+1\leqslant m\leqslant 0$ approach the values $y=1$ and $y=-1$, respectively, from inside, continuously and in a practically linear fashion. The maximum amplitude of the crest decreases monotonically and probably goes to zero as $M\rightarrow\infty$.

TABLE II.

y_2	1
c_{12}	0.195479
c_{22}	0.019673
y_{-2}	−1.879564
$c_{1,-2}$	0.199315
y_{+1}	0.637196
$c_{1,1}$	−0.004030
y_{-1}	−0.753512
$c_{1,-1}$	0.001036
y_0	−0.474998
c_{10}	0.001452
λ	0.644815

In Fig. 3a we show the flux lines (29) of the flow (48) for $M=2$. Its numerical parameters are given in Table II. The zero flux line corresponds to the profile of the free surface in the wave system of coordinates. Three less intense dipoles are localized in the regions surrounded by rectangles in the figure. These regions are scaled up in the detailed plots of Figs. 3b and 3c.

In Fig. 4 we show curves for the profile of the free surface $\eta(\theta)$ for the values $M=2$, 7, 12, and 17. Reintroducing dimensional quantities, we find that for $M=17$ and $l=100$ m we have for the crest height $h^{(34)}=l\,\eta^{(34)}(0)\approx 1.64$ m, and a velocity $c^{(34)}=\lambda^{(34)}\sqrt{gl}\approx 13.4$ m/s. The width of the crest at the level $y=0$ is of order 200 m, i.e., the criteria for smoothness (32) and weak nonlinearity (5) are easily fulfilled for all values of M.

5. EXACT SOLUTIONS OF THE FULL EQUATION IN THE QUADRATIC APPROXIMATION

Let us consider steady-state flow, whose velocity field (17) contains only one pair ($M=1$) of complex conjugate poles with multiplicity N. We will choose as a characteristic length l for the perturbation the distance of the poles from the unperturbed surface and place the pair on the vertical axis: $x_1=0$, $y_1=1$. As noted above, in this case the system of nonlinear algebraic Eqs. (24)–(27) for the variables a_{n1}, b_{n1} reduces to real form. In addition, from the symmetry condition (22) for the coefficients $G^{mm'}_{nn'}$ it follows that the diagonal elements of the matrix $V^{mm'}_{n}$, defined in (28), reduce to zero for $n=1$, i.e., $A_{1m}=B_{1m}$. Then Eq. (18) for the complex potential of the flow can be written in the form (with the label m omitted)

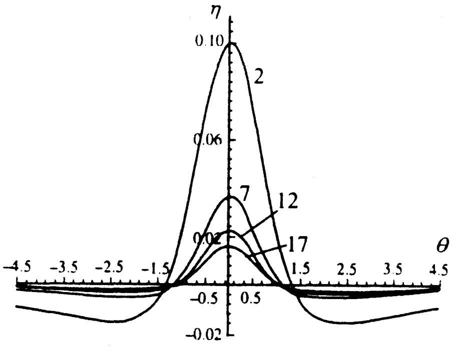

FIG. 4. Free-surface profiles of solitary waves corresponding to the solutions $\tilde{W}^{(2M+1)}_{\{N_m\}}(z)$ for $M=2$, 7, 12, 17.

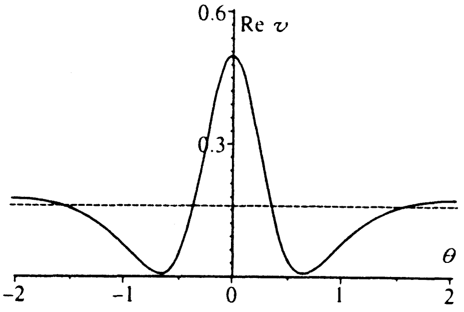

FIG. 5. Profile of the horizontal component of the velocity for the steady-state flow $W_6^{(1)}(z)$ in the wave system of coordinates.

$$W_N^{(1)}(z)=c_0z+id_0\ln\left(\frac{z-i}{z+i}\right)+\sum_{n=1}^{N-1}\left[\frac{\alpha_n c_n}{(z-i)^n}+\frac{\bar{\alpha}_n d_n}{(z+i)^n}\right], \tag{49}$$

where $c_0=A_0$, $d_0=-iA_1$, c_n, d_n are real numbers. The total number of algebraic equations in the system (24)–(26) for both cases (a) and (b) indicated in (30) equals $2N$. Numerical solutions were obtained only starting with $N\geqslant 6$ for case (a). As we noted, in this case the velocity field created by the poles of highest order has only a horizontal component at $y=0$.

The parameters of the first solution for $N=6$ are given in Table III. The distinctive feature of this solution compared with solutions of the truncated equations is the presence of a logarithmic potential (i.e., a pair of vortices located symmetrically with respect to $y=0$) and a constant flow at infinity. Figure 5 shows the dependence of the horizontal component of the flow velocity $v_1(\theta)$ at $y=0$ in the wave system of coordinates; in linear approximation, according to (8), this is proportional to the elevation $\eta^{(1)}(\theta)$. The dotted line indicates the level of this flow component to which the perturbation reduces as $|\theta|\rightarrow\infty$. There are three maxima. The amplitude of the central maximum compared to the level at infinity is $v_1(0)=0.33$, so that the criterion for weak nonlinearity is easily satisfied only at large distances from it.

6. CONCLUSION

The new class of steady-state spatially two-dimensional multipole solitary waves described in this paper, which propagate near the free surface of a deep fluid, bear a certain resemblance to the exact solutions of the canonical two-dimensional Kadomtsev–Petviashvili equation found recently by Gorshkov *et al.*,[25] which are referred to by these authors as multisolitons. However, the principal difference between their results and ours is the fact that in our case, the two-dimensional nature is due to the dependence of the field on the vertical coordinate, as a result of which our solutions are locally rotational. We have not investigated the stability of these compound structures.

There are many large-scale[10] long-lived states of the ocean surface that are well known, e.g., zones of rip tides, crowding, and Langmuir circulation, that exist during relatively peaceful weather. There are detailed descriptions of especially dangerous types of waves, e.g., storm waves of the "ninth-roller" type and tsunamis, that are smooth and of modest height in the open ocean but possess destructive force on the shores.[26,27] Although many different models have been proposed in the past, a unified theory that explains the nature of these phenomena does not yet exist.

Existing theoretical models for the excitation of waves by wind, e.g., the well-known mechanism of Miles,[28] can describe only the initial stages of development of the unstable tangential discontinuity. In this case, periodic gravitational waves grow because of a transfer of energy from the air current. Investigation of later stages of this process within the framework of the Fourier spectral method becomes ineffective, and the conclusion that the flow reduces rapidly to turbulence is unpersuasive. A variety of experimental investigations of flows with vertical velocity shifts have shown[16–18] that an intermediate stage of strongly nonlinear regular motion exists with the formation of large-scale vortex structures. The singular steady-state flows described in this paper probably reflect just these intermediate stages for the coupled atmosphere–ocean system. Their exceptional variety and weak stability due to atmospheric variability lead us to believe that such large-scale structures can indeed appear under certain conditions, and can function as a primary energy source and reason for the formation of storms, tsunamis, and other anomalous states of the surface of the open ocean. The solutions we have obtained follow from the dynamics of an ideal uniform gravitating fluid with a free surface; factors connected with stratification of the near-surface layer, the presence of a thermocline, the rotation of the Earth, etc. are of secondary importance and can only catalyze such flows.

TABLE III.

n	c_n	d_n
0	0.167829	−0.004680
1	−0.096956	−0.021142
2	0.015177	−0.063090
3	−0.007178	−0.070687
4	−0.012056	0.041017
5	−0.011781	0.011781
	$\lambda=0.539058$	

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