Theory of mode locking with a coherent absorber. I. Generation of soliton-like 2π pulses

V. V. Kozlov and É. E. Fradkin

Institute of Physics Research, St. Petersburg State University, 198904 Moscow, Russia

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We investigate a model of a laser with an absorbing cell for which the width T_{2p}^{-1} of the spectrum of the absorbing component is much narrower than the width T_{2g}^{-1} of the spectrum of the active medium. We show that it is possible to obtain stable generation of pulses of self-induced transparency by finding exact analytic solutions for a model that takes into account the effect of phase relaxation in the absorber, linear losses in the resonator, and saturation of the gain. In this model the amplifier is treated as a slow medium, i.e., $\tau_p \ll T_g$. We note the decisive role played by an optical filter that limits the gain bandwidth in ensuring the stability of the mode-locking regime against the appearance of a regime of cw emission. We discuss the choice of optimal parameters for the absorbing and amplifying media that make experimental realization of a regime of stable mode locking with a coherent absorber easier.

1. INTRODUCTION

From the earliest days of laser engineering, researchers have concentrated much of their effort on creating stable sources of short, and especially ultrashort, optical pulses. A broad review that encompasses the experimental and theoretical aspects of the generation of ultrashort pulses can be found in the paper by New.[1] The development of techniques for passive mode locking in the colliding-pulse regime[2,3] and use of compression of phase-modulated pulses in a dispersive medium has led to further progress in the direction of femtosecond-duration pulses from dye lasers.[4,5] An analogous mechanism forms the basis of the method of generating ultrashort pulses in solid-state lasers now known under the name of mode locking by an auxiliary pulse (MLAP).[6,7] The basis of the MLAP method is the use of phase self-modulation to bring about coherent addition of a laser pulse and a chirp pulse in an external nonlinear resonator.[8] Very recently, Spence *et al.*[9] proposed to use the induced Kerr lens within the amplifier as an effective mechanism for mode locking (KLML). Stable generation of pulses is achieved because the losses for this regime are lower than they are in the regime of cw oscillation.[10] The difference in losses is caused by self-focusing in the nonlinear amplifier.[10,11,12] Both techniques, MLAP and KLML, are based on soliton generation, which is described by the nonlinear Schrödinger equation.[13]

In this paper we propose another method for generating pulses that is based on the phenomenon of self-induced transparency (SIT).[14,15] In 1967, McCall and Hahn observed that an absorber with an inhomogeneously broadened line shape becomes transparent (neglecting phase relaxation) to pulses that exceed certain characteristic power and duration thresholds (i.e., the area under the field envelope of the pulse should be larger than π). The initial stage of pulse propagation is accompanied by establishment of a stationary shape for the envelope in the form of the function $\mathrm{sech}(x)$, while the pulse area approaches a value equal to 2π. The transparency mechanism is based on the coherent interaction of radiation with the absorber: the leading edge of the pulse converts atoms of the absorber to the upper excited state, while the trailing edge of the pulse induces stimulated emission from the upper state that returns energy to the pulse field. This coherent exchange of energy between the medium and the field causes an appreciable delay of the pulse, whose velocity in the medium can differ by several orders of magnitude from its velocity in vacuum. It is noteworthy that almost all the essential features of SIT can occur when a pulse interacts with an absorber without inhomogeneous broadening as well.

The simplest and most natural way to generate SIT pulses is to place a cell containing the absorbing medium within the laser resonator. In order to ensure coherence of the interaction of the pulse with the absorber, a wideband amplifier must be chosen such that $T_{2p}^{-1} \ll \tau_p^{-1} \ll T_{2g}^{-1}$, where T_{2g} and T_{2p} are the phase memory times for the amplifier and the absorber. The main problem in achieving stable mode locking is to suppress the regime of cw oscillation. A necessary condition for this is the requirement that the overall gain of a weak signal be negative throughout the band of frequencies at the leading and trailing edges of the pulse. The narrower the spectral line profile of the absorber, the more difficult it is to satisfy this condition, which requires an appropriate increase in the concentration of absorbing atoms. However, increasing the concentration of absorbers brings a second, competitive process comes into play—the loss of energy within an absorber can become so large that it leads to the collapse of pulse generation. We will show that it is possible to choose a compromise absorber density only in the presence of an optical filter. The filter prevents shortening of the pulse, while also eliminating the possibility that a regime of cw oscillation will arise in the wings of the line profile.

The coherence of the pulse field interaction with the absorber can cause another instability mechanism to develop—instability with respect to small amplitude modulations. We solved a similar problem with regard to stability in Ref. 16; here, we pause only to discuss the essential results. We define a stability parameter equal to the square of twice the ratio of the Rabi frequencies for the amplifier and absorber:

$$s=\left(2\frac{R_g}{R_p}\right)^2,\quad R_g\equiv\frac{2d_g}{\hbar}\mathscr{E},\quad R_p\equiv\frac{2d_p}{\hbar}\mathscr{E}, \tag{1}$$

where d_g and d_p are the amplifier and absorber dipole moments. In these expressions, we may pick $\mathscr{E}$ to be the pulse amplitude; however, the field we choose is not important, since the ratio of Rabi frequencies enters into (1) and the field cancels. The condition for stable propagation is that $s\geqslant 1$.

When both stability conditions are satisfied, it is possible to generate SIT pulses with area under the envelope equal to 2π (for the absorbing medium) and with considerable time delay for a single round trip through the resonator. From these characteristic properties we conclude that the absorber plays a primary role in the process of pulse generation. In fact, the role of the amplifier reduces to compensating for incoherent losses in the absorber and linear field losses in a resonator. An appropriate formalism in which to discuss the theory of SIT is the theory of a two-component medium. A new physical effect appears that is not contained in the theory of McCall and Hahn—the pulse amplitude (and thus the duration) is uniquely determined. Recall that in the classic paper by McCall and Hahn,[15] the amplitude is a function of the energy of the input pulse and its transient evolution in the medium.

In this paper we propose to present our material in the following order. In Sec. 2 we set forth the basic equations for the model, and justify the approximations used. In this section we write the polarization of the amplifying medium in the form of a series with respect to energy and field intensity. In Sec. 3 we obtain a solution in the form of 2π pulses and discuss the influence of the filter and dispersion of the gain on the soliton parameters. We analyzed the transient processes by which the steady-state field shape is set up in the resonator, and derived stability conditions in Ref. 16.

2. THE MODEL AND BASIC EQUATIONS

The model we propose is based on the simultaneous solution of the wave equation and two systems of Bloch equations for the amplifier and the absorber. The description of the interaction of the field with the amplifying medium can be simplified by expanding the polarization in the small parameter T_{2g}/τ_p. We also include the effect of an optical filter which limits the gain bandwidth. We are interested in a regime in which a sequence of short pulses is generated with repetition rate equal to the time for one round trip of the resonator. We will assume that a rather large number of modes participate in the generation, so that we may neglect the spatial extent of each pulse compared to the resonator length. This allows us to replace the periodic boundary conditions on the field

$$E(t+\tau_{cav},z+l_{cav})=E(t,z),$$

where l_{cav} and τ_{cav} are the length of the resonator and time for a circuit of the resonator, by the condition at infinity

$$E\left(u=\frac{t-z/v_p}{\tau_p},z\right)\xrightarrow[z\to\pm\infty]{}0.$$

The solution to the problem is obtained in the form of a pulse with steady-state shape $E(u)$, duration τ_p, and moving at velocity v_p. This approximation is valid if the relative change in the field of the pulse is small during its passage through any of the resonator elements. We will assume that the dynamics of interaction of the radiation with both media is determined by the following conditions: both before a pulse arrives at a given point in the resonator and after it passes that point, the losses in the absorber return to their original levels via spontaneous relaxation. Furthermore, within the same interval of time, the initial value of the gain is re-established in the process of pumping. In order for this to occur, it is necessary to choose an active medium with a recovery time (i.e., the lifetime of the upper level) that is comparable to the round trip time of the resonator. As a rule, this condition is fulfilled for dye and semiconductor lasers.

We will assume that the spectral line widths of the absorber and amplifier satisfy $\Delta\omega_g\gg\Delta\omega_p$. When this condition holds, a pulse formed by mode locking can interact coherently with the absorber, since $\tau_p<T_{2p}$. In the similar formulation of this problem given in Ref. 17, a solid-state active medium was used as the amplifier and dynamic saturation was neglected; without this latter effect, stable generation of SIT pulses is impossible. The formalism we use in the present paper was first introduced by us in Ref. 18 and 19; however, in that paper we made a sign error in deriving the stability conditions, which led to incorrect results. In what follows, we give a correct and expanded theory of mode locking with a coherent absorber.

Let us write the wave equation in the slowly varying amplitude and phase approximation, first separating the total field into real amplitude and phase. We also separate the polarizations of the amplifying and absorbing media into inphase and quadrature components. We will assume that the line profiles of both media are homogeneously broadened:

$$\left\{\left(\frac{\partial}{\partial z}-\frac{v_p^{-1}-c^{-1}}{\tau_p}\frac{\partial}{\partial u}\right)+L_{BW}\right\}E=-\frac{L_g}{T_{2g}}Q_g-\frac{L_p}{T_{2p}}\mu^{-1}Q_p, \tag{2a}$$

$$E\left(\frac{\partial}{\partial z}-\frac{v_p^{-1}-c^{-1}}{\tau_p}\frac{\partial}{\partial u}\right)\varphi=\frac{L_g}{T_{2g}}P_g+\frac{L_p}{T_{2p}}\mu^{-1}P_g. \tag{2b}$$

In this case the total field and polarization can be written in the form:

$$\mathscr{E}(z,t)=\frac{\hbar}{dg}E(u,z)\exp\{i[\varphi(z,u)-(\omega_0t-kz)]\}, \tag{3a}$$

$$\mathscr{P}(z,t)=\{d_gn_g[P_g(u,z)+iQ_g(u,z)]+d_pn_p[P_p(u,z)+iQ_p(u,z)]\}\exp\{i[\varphi(z,u)-(\omega_0t-kz)]\}. \tag{3b}$$

In order to identify the basic features of pulse generation and to simplify the exposition, we will assume that the centers of the amplification and absorption lines, and the transmission maximum of the filter L_{BW}, all occur at the same frequency ω_0. We will look for steady-state (i.e., depending only on the wave coordinate u) non-phase-modulated solutions for the field in the form of pulses with a carrier frequency ω_0. In these equations we have retained the coordinate z and phase $\varphi(z,u)$ for further use in deriving conditions for stability. We have introduced the following notation: n_g and n_p are the concentrations of active and passive atoms; T_{2g} and T_{2p} are the times for transverse relaxation of the amplifier and absorber (if we neglect inhomogeneous broadening, the inverse values of these quantities are the spectral line widths); and d_g and d_p are the dipole transition moments for the amplifier and the absorber. $L_{BW}(\omega)$ is the value of the linear losses caused by the optical filter, and

$$L_g=\frac{2\pi\omega_0 d_g^2 n_g}{c\hbar}T_{2g},\quad L_p=\frac{2\pi\omega_0 d_p^2 n_p}{c\hbar}T_{2p},$$

$$\mu=\frac{d_p}{d_g}.$$

In order to determine the polarization components on the right sides of Eqs. (2), we write two complete systems of Bloch equations for the absorber and amplifier:

$$\frac{\partial}{\partial u}P_g=\frac{\partial\varphi}{\partial u}Q_g-\frac{P_g}{T}, \tag{4a}$$

$$\frac{\partial}{\partial u}Q_g=-\frac{\partial\varphi}{\partial u}P_g-\frac{Q_g}{T}-\tau_p E N_g, \tag{4b}$$

$$\frac{\partial}{\partial u}N_g=\tau_p E Q_g; \tag{4c}$$

$$\frac{\partial}{\partial u}P_p=\frac{\partial\varphi}{\partial u}Q_p-\frac{P_p}{T\kappa}, \tag{5a}$$

$$\frac{\partial}{\partial u}Q_p=-\frac{\partial\varphi}{\partial u}P_p-\frac{Q_p}{T\kappa}-\tau_p\mu E N_p, \tag{5b}$$

$$\frac{\partial}{\partial u}N_p=\tau_p\mu E Q_p. \tag{5c}$$

Here we have introduced the additional notation

$$T=T_{2g}/\tau_p,\quad \kappa=T_{2p}/T_{2g}. \tag{6}$$

In Eqs. (4c) and (5c) we have omitted terms responsible for the relaxation of the population differences N_g and N_p. This approximation is valid when the duration of a pulse τ_p is much shorter than both times T_{1g} and T_{1p}, and the dynamics of pulse generation is not related qualitatively to the effects of spontaneous relaxation. Nevertheless, we implicitly take into account relaxation of N_g and N_p to their stationary values when we speak of the mutual independence of pulses separated by time interval τ_{cav}.

The form of the optical filter transmission function L_{BW} can be approximated by various functions, e.g., exponential, as was done in Ref. 20, or Lorentzian, as in Ref. 21. For our purposes it is convenient to choose a parabolic dependence

$$L_{BW}(\omega)=L_{cav}\left[1+\left(\frac{\omega-\omega_0}{\Delta\omega_p}\right)^2\right], \tag{7}$$

where L_{cav} are the linear losses of the resonator.

Let us take a Fourier transform; then the transfer function of the filter in the time representation is

$$(\omega-\omega_0)\to -i\frac{\partial}{\partial t},\quad L_{BW}(t)=L_{cav}\left[1-\frac{1}{\Delta\omega_f^2}\frac{\partial^2}{\partial t^2}\right]. \tag{8}$$

The combination of the optical filter $L_{BW}(t)$ and the dispersion of the gain constitutes an effective filter that bounds the spectral width of the pulse.

It is advisable to use the smallness of the ratio $T=T_{2g}/\tau_p$ to simplify the expressions for the polarization components P_g and Q_g rather than solving the Bloch equations (4). Our procedure for reducing the Bloch equations (4) is analogous to the formulas developed in Ref. 22, where the polarization was written as a function of the amplitude, its derivatives, and the pulse energy by expanding the components of the Bloch vector in the small parameter $(\delta\cdot\tau_p)^{-1}$, where δ is the offset of the pulse carrier frequency from the center of the amplification (absorption) line shape. We will obtain an analogous expansion by replacing δ with T_{2g}^{-1}.

Once more it is convenient to turn to complex variables, using the formulas

$$p(u,z)=\frac{1}{T_{2g}}P_g(u,z)\exp[i\varphi(u,z)],$$

$$q(u,z)=\frac{1}{T_{2g}}Q_g\exp[i\varphi(u,z)], \tag{9}$$

$$e(u,z)=E\exp[i\varphi(u,z)].$$

From Eq. (4) we find the solution for the polarization components:

$$p(u,z)-iq(u,z)=\frac{i}{T}\int_{-\infty}^{u}e(\bar{u},z)N_g(\bar{u},z)\times\exp\left[-i\frac{u-\bar{u}}{T}\right]d\bar{u}, \tag{10}$$

making the change of integration variable $v=u-\bar{u}$, we obtain

$$p(u,z)-iq(u,z)=\frac{i}{T}\int_0^{\infty}e(u-v,z)N_g(u-v,z)\times\exp\left(-\frac{v}{T}\right)dv. \tag{11}$$

In writing expressions (10) and (11) we have taken into account the initial condition for the polarization of the medium:

$$p(-\infty,z)=q(-\infty,z)=0.$$

Assuming that the components of the Bloch vector and field envelope vary slowly compared to the quantity v/T, we obtain the Taylor series

$$p(u,z)-iq(u,z)=i\sum_{n=0}^{\infty}(-1)^n T^n\frac{\partial^n}{\partial u^n}\times[e(u,z)N_g(u,z)]. \qquad (12)$$

For the purposes of this paper it is sufficient to keep the first three terms of the expansion (12), up to $n=2$ inclusively. The explicit inversion in explicit series form is

$$N_g(u,z)=T^0N_0(u,z)+T^1N_1(u,z)+T^2N_2(u,z). \qquad (13)$$

Substituting the expansions (12) and (13) into the equation for the inversion (4.3), and once more returning to real variables, we find:

$$N_0=\text{const}, \qquad (14a)$$

$$N_1=-N_0\tau_p J(u,z), \qquad (14b)$$

$$N_2=\tfrac{1}{2}N_0\tau_p^2[J^2(u,z)+E^2(u,z)]. \qquad (14c)$$

Here

$$J(u,z)=\int_{-\infty}^{t}E^2(\bar{t},z)d\bar{t}=\tau_p\int_{-\infty}^{u}E^2(\bar{u},z)d\bar{u} \qquad (15)$$

is the energy that has passed through the amplifier by time t. The constant component of the inversion $N(u=-\infty,z)=N_0$ is determined by the pump amplitude and can be set equal to $+1$ without loss of generality. Also, in deriving Eqs. (14b), (14c) we made use of the initial conditions for N_1 and N_2:

$$N_1(u,z)|_{u\to-\infty}=N_2(u,z)|_{u\to-\infty}=0.$$

Having done this, we can obtain the desired expressions for the polarization components:

$$\frac{1}{T_{2g}}Q_g(u,z)=-E+T\left[\frac{\partial}{\partial u}E+\tau_p JE\right]-T^2\left[\frac{1}{2}\tau_p^2J^2E+\frac{3}{2}\tau_p^2E^3+\tau_p J\frac{\partial}{\partial u}E+\frac{\partial^2}{\partial u^2}E-E\left(\frac{\partial\varphi}{\partial u}\right)^2\right], \qquad (16a)$$

$$\frac{1}{T_{2g}}P_g(u,z)=-TE\frac{\partial\varphi}{\partial u}+T^2\left[\tau_p JE\frac{\partial\varphi}{\partial u}+2\frac{\partial\varphi}{\partial u}\times\frac{\partial E}{\partial u}+E\frac{\partial^2\varphi}{\partial u^2}\right]. \qquad (16b)$$

The first and second terms of (16a) correspond to linear amplification of the field amplitude during its passage, taking into account possible carrier frequency offsets from the line center; accordingly, the third and fourth terms correspond to gain saturation. The second term in (16a) corresponds to the round trip time delay of the resonator, which is associated with linear dispersion; changes in the value of the time delay due to the effects of saturation are given by the sixth term. In accordance with the widely used terminology, we will speak of the amplifier as "slow" when the recovery time of the medium is much longer than the pulse duration. When the relations between these times are reversed, we use the term "fast." It is interesting that (16a) contains a term proportional to the cube of the amplitude, which corresponds to the response of a fast medium. However, in this case our physical interpretation of this term is not based on the rate-equation approximation, where it is responsible for saturation of the medium. This is already clear from the fact that the sign of the cubic term in the field the opposite of what we might expect if it were simply a consequence of the instantaneous response to the applied field. We may assume that this is a manifestation of coherent effects in the amplifier. In the papers like those of by Ref. 23 and 24, where passive mode locking was discussed, there is usually no mention of this type of composite response of a slow medium. Usually it is assumed[24] that the fast processes depend on the field intensity E^2, while the slow processes depend on the pulse energy $J(u,z)$. The influence of the coherent component needs to be evaluated for each specific case. For our model we find that its inclusion is important for values of the stability parameter s close to unity. Finally, the term in (16a) that contains the first derivative of the field amplitude is responsible for limiting the gain bandwidth, and plays the role of a filter.

The first and second terms in (16b) determine pulling of the carrier frequency of the pulse when the latter does not coincide with the maximum of the gain coefficient, along with variation of the magnitude of the pulling due to the saturation effect. The last two terms in (16b) are associated with dispersion of the group velocity.

A necessary condition for the correctness of the expansion (16a) and (16b) is the assumption that T is small. Furthermore, it is also necessary to check that $T\tau_p J<1$. Equations (16) are derived under the assumption that the phase $\varphi(u,z)$ varies slowly; therefore, they are valid for small offsets from resonance.

3. SOLUTION IN THE FORM OF SOLITON-LIKE SIT PULSES

In the previous sections we derived the expressions for in-phase and quadrature portions of the amplifier polarization in detail. If we are interested only in non-phase-modulated solutions for the field, we set $P_g=0$. In the expression for Q_g, we keep only the two leading terms of the expansion (up to T^1 inclusively) and substitute them into the equation for the field amplitude (2a). In this approximation we limit ourselves to including only the first term in the expansion for the transmission function of the filter (8), i.e., we assume that the losses are constant over the entire band of frequencies occupied by the pulse spectrum. It is not difficult to verify that a solution in the form

$$E_0=A_0/\cosh u \qquad (17)$$

satisfies the system of Eqs. (2a), (5), and the usual boundary conditions, if the following three algebraic relations for the pulse parameters v_p, τ_p, and A_0 are fulfilled:

$$\left(\frac{\theta}{\pi}\right)^2=s, \qquad (18)$$

$$\frac{v_p^{-1}-c^{-1}}{L_g\tau_p}=3\kappa T\left(\eta-\frac{1}{2}T_{2g}J_0\right)+T-\frac{1}{2}T_{2g}J_0, \qquad (19)$$

$$\left(\eta-\frac{1}{2}T_{2g}J_0\right)(3T^2+4\kappa^{-1}T+\kappa^{-2})=\rho\kappa^{-2}. \quad (20)$$

We have introduced the following notation:

$$\theta\equiv\int_{-\infty}^{+\infty}E\,dt=\pi A_0\tau_p$$

—Pulse area (21)

$$J_0\equiv\int_{-\infty}^{+\infty}E^2dt=J(\infty)=2A_0^2\tau_p$$

—Total pulse energy (22)

$$\eta=\frac{L_g-L_{cav}}{L_g},\quad \rho=\frac{L_p}{L_g}. \quad (23)$$

The relation between the relaxation times $T_{2g}\ll T_{2p}$ creates a precondition for self-induced transparency. Let us demonstrate this. Let us use the dipole moment for the gain medium d_g to normalize the field in Eq. (3a); then Eq. (18) determines the area under the field envelope with regard to propagation in the amplifier, and equals $\sqrt{s}\pi$. At the same time, for the absorbing component this same pulse has area 2π. The fact that the area under the envelope is constant, and that it is independent of the parameters of the problem for propagation of a pulse in an absorbing medium, is a consequence of the coherent interaction of the field with the absorber. We note a still more obvious similarity with SIT if we write Eq. (19) in the form:

$$\frac{c}{v_p}=1+\frac{3(\Omega_pT_{2p})^2}{3(\kappa T)^2+4\kappa T+1}+(1-s)(\Omega_gT_{2g})^2, \quad (24)$$

where

$$\Omega_p^2=2\pi\omega_0d_p^2n_p/\hbar,\quad \Omega_g^2=2\pi\omega_0d_g^2n_g/\hbar$$

are the squares of the cooperative frequencies for the absorbing and amplifying media respectively. In Ref. 16 we showed that if the dominant interaction of the field with the resonator medium is coherent interaction with the absorber (i.e., the amplification and dissipation processes can be considered to be perturbations), then stable pulse generation requires that $s\gg1$. Furthermore, as is clear from Eq. (24), the amplifier forces the pulse to speed up. One of the basic manifestations of SIT is the considerable delay exhibited by the pulse as it propagates in the absorber, due to the dynamic exchange of energy between the field and medium.[25] The second term on the right side of Eq. (24) is associated with slowing of the pulse in the absorber. For $\tau_p\ll T_{2p}$ (or $\kappa T\gg1$), Eq. (24) reduces to the well-known expression for the velocity of a 2π pulse[15]

$$\frac{c}{v_p}=1+(\Omega_p\tau_p)^2. \quad (25)$$

From Eq. (20) we determine the duration of the generated pulse. The graphical solution to this equation is illustrated in Fig. 1. The lower (dashed) branch corresponds to an unstable solution to Eq. (20). This unphysical solution is characterized by increasing pulse generation power as the density of the absorber increases. Our use of the term

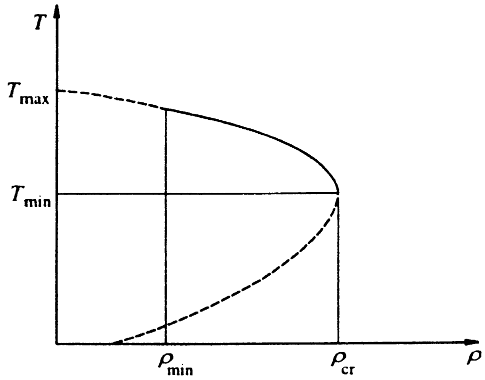

FIG. 1. Dependence of signal power on the density of absorbers. The dashed curves correspond to the region of unstable pulse generation.

"power" for the quantity T is validated by the presence of a strict relation between A_0 and τ_p, determined by Eq. (18); therefore the quantity T is directly proportional to the increase in J_0.

To obtain a regime of stable soliton-like pulses, it is necessary to ensure that the overall gain of a weak signal at the leading and trailing edges of the pulse is negative. In Ref. 16 we showed that this condition is satisfied when

$$\rho>\frac{\eta^2}{4}\frac{\kappa^2}{\kappa_f^2}\quad\text{for }\kappa_f^2\gg1, \quad (26)$$

$$\kappa_f^2\equiv1+\frac{1-\eta}{(T_{2g}\Delta\omega_f)^2}. \quad (27)$$

The requirement on the density of the absorber imposed by (26) determines the lower threshold for ρ. To the left of ρ_{min}, the regime of generation of 2π pulses is found to be unstable against the appearance of a regime of cw oscillation.

The characteristic values indicated in Fig. 1 are determined by expressions given below, where T_{min} is given by

$$T_{min}=\frac{\kappa^{-1}}{3}\left[\left(\eta\frac{\kappa}{s}-\frac{L}{3}\right)+\sqrt{\left(4\eta\frac{\kappa}{s}-1\right)+\left(\eta\frac{\kappa}{s}-\frac{4}{3}\right)^2}\right]. \quad (28)$$

We will be interested in the generation of short pulses, which requires that $4\eta\kappa/s\gg1$. Then the expressions for the characteristic values shown in Fig. 1 simplify:

$$T_{min}=\frac{2}{3}T_{max},\quad T_{max}=\frac{\eta}{s}, \quad (29)$$

$$\rho_{min}=\left[\frac{1}{2}\eta\frac{\kappa}{\kappa_f}\right]^2,\quad \rho_{cr}=\eta\left[\frac{2}{3}\eta\frac{\kappa}{s}\right]^2. \quad (30)$$

In order to obtain stable generation of 2π pulses it is necessary to fulfill the obvious condition

$$\rho_{min}<\rho_{cr},\quad\text{i.e. }\eta\left(\frac{4}{3}\frac{\kappa_f}{s}\right)^2>1. \quad (31a)$$

The more rigorously inequality (31a) is satisfied, the wider the region $\rho_{\min}<\rho<\rho_{cr}$ within which we may choose the absorber density. A narrow optical filter (larger than κ_f^2) ensures a wide region of stability due to the limits on the gain bandwidth.

If the optical filter is very wide, i.e., κ_f^2 is close to unity, then the expression for $\rho_{\min}$ (26) becomes unsuitable and must be replaced by

$$\rho_{\min}=(1-\sqrt{1-\eta})\kappa^2 \quad \text{for } \kappa_f^2=1,$$

(see Ref. 16). Accordingly, condition (31a) becomes

$$1-\sqrt{1-\eta}<\frac{4}{9}\frac{\eta^3}{s^2}. \tag{31b}$$

The bound on the value of the stability parameter s, i.e., $s\geqslant 1$, makes it impossible to satisfy (31b) for any pump value. We are led to an important conclusion: stable generation of 2π pulses is impossible in the absence of a band-limiting filter.

Let us also note that the duration of the 2π pulses remains finite even if we do not include the boundedness of the gain bandwidth (29). That is, the coherent absorber itself fulfills the role of a filter. The mechanism for this is rather simple. The phenomenon of gain saturation bounds the growth of the generated pulse energy. The fixed relation between the duration and amplitude of a pulse implied by the constancy of the area (18), in turn, limits the duration of the pulse. Of course, gain dispersion and finiteness of the bandwidth of the optical filter contribute to the narrowing of the pulse spectrum. These effects are not taken into account in deriving Eqs. (18)–(20); therefore, Eq. (29) give saturated values for the quantity T. The theory can be further refined by including the third terms in the expansion of the polarization components Q_g and P_g, (16a) and (16b), which are proportional to T^2. At the same time we need to include the second term in the expansion for the transmission function of the filter (8).

Rather than develop the theory further, we note that the solution in the form of (17) satisfies Eqs. (2a), (5), along with the usual boundary conditions, for relations between the pulse parameters v_p, τ_p, and A_0 other than (18)–(20):

$$\left(\frac{\theta}{\pi}\right)^2=\frac{1}{4}s,\quad \frac{c}{v_p}=1+\left(1-\frac{s}{2}\right)(\Omega_g T_{2g})^2,$$

$$\left(\eta-\frac{1}{2}T_{2g}J_0\right)(1+\kappa T)=\rho. \tag{32}$$

This alternative solution consists of a π pulse for the absorber moving with the velocity of light in the absorbing component. We will not pause to analyze the properties of this solution in detail. It would seem that it is obtained over a wide interval of durations for $\tau_p\approx T_{2p}$. For smaller values of the duration the coherent interaction of the pulse with the medium predominates, and in accordance with the classical properties of a π pulse it passes through the absorber, which leaves it inverted. The detailed nature of the propagation creates the precondition for instability at the trailing edge of the pulse, where the medium turns out to be amplifying for a weak signal. Furthermore, it is doubtful that this regime is energetically advantageous for pulse generation.

There is also another reason that allows us to ignore solutions in the form of π pulses within the framework of our paper. The generation regime for 2π pulses that we have investigated also includes an analysis of stability; hence, no other regimes can compete with it in the region where it is stable.

4. 2π PULSES TAKING INTO ACCOUNT DISPERSION OF THE GAIN

A distinctive feature of the theory of passive mode locking with a coherent absorber is distortion of the hyperbolic-secant shape of the pulse when the effect of the optical filter and dispersion of the gain is included, in contrast to the theory for fast and slow absorbers.[20–22] Thus, when we substitute the full expressions (16a) and (16b) for the polarization components Q_g and P_g, including terms proportional to T^2, into Eqs. (2a) and (2b), we are no longer able to find analytic solutions to the problem. However, we can achieve a considerable simplification by identifying the primary interaction. For short enough pulses, with $\tau_p\ll T_{2p}$, the coherent absorber plays the primary role in forming the field envelope, while the role of the amplifying medium reduces to compensating losses generated by the optical filter and relaxation processes in the absorber. Let us write the equation for the field in the form

$$-\frac{v_p^{-1}-c^{-1}}{L_g\tau_p}\dot{E}+\frac{\rho}{\mu T_{2p}}(Q_p)_{\rm sol}=-\frac{\rho}{\mu T_{2p}}(Q_p)_{\rm loss}-\frac{Q_g}{T_{2g}}$$
$$+(\eta-1)\left[1-\frac{1}{(\Delta\omega_f\tau)^2}\frac{\partial^2}{\partial u^2}\right]E. \tag{33}$$

The polarization of the absorbing medium has been split into two parts: $(Q_p)_{\rm sol}$, which is obtained by assuming $T_{2p}\to\infty$ and is thus responsible for the formation of a classic SIT pulse, and $(Q_p)_{\rm loss}$, which contains the rest of the polarization Q_p, taking into account the finiteness of T_{2p}. The coherent absorber plays the primary role in the formation and propagation dynamics of the pulse, and specifically the soliton part of the polarization $(Q_p)_{\rm sol}$. The energy loss and distortion of the envelope connected with the nonsoliton part of the polarization $(Q_p)_{\rm loss}$ decreases as the pulse gets shorter, in the ratio τ_p/T_{2p}. We will assume that linear losses in the resonator and amplifier also play a secondary role in forming the field profile. We will make some quantitative estimates of this approximation at the end of the section.

Let us use the smallness of the right-hand side of Eq. (33) and solve (33) first, replacing the expression on the right-hand side by zero. We then obtain a solution in the form of a classic SIT pulse with an envelope (17) and the parameters

$$\left(\frac{\theta}{\pi}\right)^2=s,\quad \frac{c}{v_p}=1+(\Omega_p\tau_p)^2. \tag{34}$$

In the theory of SIT, the amplitude A_0 and the associated duration τ_p remain arbitrary and are determined by the parameters of the input pulse and transient processes in the system.[15] This behavior is due to the conservation properties of this system of equations. Amplification and absorption are taken into account as perturbations. The condition for solvability of the inhomogeneous Eq. (33) is orthogonality of the right side to the solution of the unperturbed equation. By separating the soliton and nonsoliton parts in the expression for the inversion in the absorber, as we did for the polarization, we obtain

$$\frac{\rho}{\mu^2 T_{2p}} (N_p)_{loss} \int_{-\infty}^{+\infty} + \frac{N_g}{T_{2g}} \int_{-\infty}^{+\infty} + (1-\eta) \times \left[J_0 + \frac{\tau_p}{(\Delta\omega_p \tau_p)^2} \int_{-\infty}^{+\infty} \dot{E}_0^2 du \right] = 0. \tag{35}$$

Expressions (34) and Eq. (35) clarify the physics of the approximation we are using. The velocity of the pulse, the shape of the envelope and the area under it are determined by the coherent properties of the absorber. The amplifying medium compensates for linear losses in the absorber and leads to generation of pulses with fixed energy. The band-limited losses limit the pulse shortening. In fact, Eq. (35) can be written in the form of the law of conservation of energy.

In the linear approximation $(N_p)_{loss} \ll (N_p)_{sol}$, we have the following equation for $(N_p)_{loss}$:

$$\frac{\partial}{\partial u} (N_p)_{loss} - \frac{\ddot{E}_0}{\dot{E}_0} (N_p)_{loss} = -4(\kappa T)^{-1} \frac{E_0}{\dot{E}_0} \frac{1}{A_0^2} \int_{-\infty}^{u} \dot{E}_0^2 du. \tag{36}$$

From this we obtain at once

$$(N_p)_{loss} \Big|_{-\infty}^{+\infty} = \tfrac{8}{3} (\kappa T)^{-1}. \tag{37}$$

The components entering into (14) do not contain enough information to find an expression for the inversion of the active medium, so we must consider the next term in the expansion. Then the inversion is

$$N_g(u) = 1 - T_{2g}J + \frac{1}{2}(T_{2g}J)^2 - \frac{1}{6}(T_{2g}J)^3 + \frac{1}{2} T_{2g}^2 \times E_0^2(u) - T_{2g}^3 \tau_p \int_{-\infty}^{u} E_0^4(\bar{u}) d\bar{u} + T_{2g}^3 \tau_p^{-1} \times \int_{-\infty}^{u} \dot{E}_0^2 d\bar{u}. \tag{38}$$

Finally, we find that the transmitted pulse empties the upper level by an amount:

$$N_g(u) \int_{-\infty}^{\infty} = -(T_{2g}J_0) \left[1 - \frac{1}{2} T_{2g}J_0 + \frac{1}{6}(T_{2g}J_0)^2 + \frac{2}{3} sT^2 - \frac{1}{3} T^2 \right]. \tag{39}$$

Substituting Eqs. (37) and (39) into the conservation law (35), we obtain an equation for the energy (duration) of the pulse:

$$3T^2 \left[\eta - sT - \frac{1}{3} T^2 (\kappa_f^2 - 2s - 2s^2) \right] = \rho \kappa^{-2}. \tag{40}$$

For $T \gg \kappa^{-1}$, Eq. (20) reduces to Eq. (40) without including the next term in curly brackets. This sequence of operations confirms the validity of the approximations we have used.

The most significant difference between Eqs. (20) and (40) is the presence of the next term in curly brackets in (40). It changes the duration of the pulses generated, but does not change the qualitative form of the dependence of the pulse duration on the density of the absorber shown in Fig. 1. The values of characteristic points change in the following way:

$$T_{min} = \sigma_{min} \left(\sqrt{1 + \frac{4}{3}\frac{\eta}{3} \sigma_{min}^{-1}} - 1 \right),$$

$$\sigma_{min} = \frac{9s}{8(\kappa_p^2 - 2s - 2s^2)}, \tag{41}$$

$$T_{max} = \sigma_{min} \left(\sqrt{1 + 2\frac{\eta}{s} \sigma_{max}^{-1}} - 1 \right),$$

$$\sigma_{min} = \frac{3s}{2(\kappa_p^2 - 2s - 2s^2)}, \tag{42}$$

$$\rho_{min} = \left(\frac{1}{2} \eta \frac{\kappa}{\kappa_f} \right)^2, \quad \rho_{cr} = \frac{3}{2} \left(\eta - \frac{s}{2} T_{min} \right) T_{min}^2 \kappa^2. \tag{42}$$

In Eqs. (41) we see a competition between two processes: a narrowing of the pulse spectrum caused the effective optical filter ($\propto \kappa_f^2$), and, conversely, a broadening of the spectrum due to the more efficient removal of the amplifier inversion ($\propto s^2$) and to coherent processes in the amplifier ($\propto s$). Here we are dealing with the coherent component, which responds to the instantaneous field intensity and opposes saturation of the fast medium. In other words, the coherent properties are manifest in a slowing down of amplifier saturation. The use of the term "coherent" can be justified by starting from analysis of the expression for the amplifier inversion (38). The next term in (38) is different from zero only within a narrow interval of time ($\approx \tau_p$) and can in no way change the value of the inversion after the pulse passes through it. The expansion in a series of the components of the Bloch vector for the amplifying medium (12), (14) does not include (recovery) processes that relax the population difference ($\sim T_{2g}$) to its equilibrium value under the action of the pump. Therefore, the term proportional to the instantaneous field intensity can be only a manifestation of coherent exchange of energy between the amplifying medium and the field of the pulse, which take place without energy loss.

If of these two competing processes it is the effect of decreased pulse duration that predominates, then there exists a small parameter range for which the expression in (41) under the root sign is negative. We assume that no real physical process is connected with this. Its explanation lies in the insufficient accuracy resulting from our inclusion of only the first three terms in the expansion for the amplifier polarization. We can obtain real values of the pulse duration if we consider the next orders in the expansion of $Q_g(u)$.

Equation (41) allows us to establish the limits of applicability of the theory without taking into account dispersion

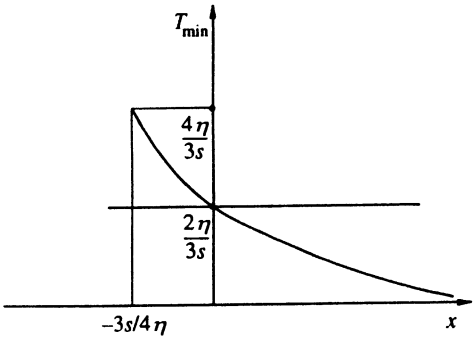

FIG. 2. Dependence of generated pulse power (for an absorber density $\rho=\rho_{cr}$) on the width of the filter $\varkappa_f^2$. Here we have $X=8/9s(\kappa_f^2-2s-2s^2)$. The straight line corresponds to a theoretical analysis without including the effects of the filter and dispersion of the gain.

of the gain. The theory accurately describes the dynamics of pulse generation when the action of the filter and the "additional saturation" cancel each other:

$$\kappa_f^2-2s(1+s)=0. \tag{43}$$

Deviations from Eq. (43) toward positive values indicate that the effective filter predominates and the duration of the pulse is increased compared to (20). A change in the parameter S influences the duration of the pulse in two ways. First, it increases s because the gain saturation coefficient leads to an increase in the pulse duration; second, pulse shortening due to the additional saturation also takes place as s increases, see (43). This second effect is considerably less important. In Fig. 2 we show the dependence of $T_{\min}$ on the width of the filter.

In order to get an idea of the size of the parameters that are needed to achieve stable pulse generation, we turn to Fig. 3, where the crosshatched region contains the stable parameters $s(\eta)$ for a filter width that satisfies Eq. (43). Then the choice of allowed values of the absorber density lies in the interval

$$\frac{1}{4}\frac{\eta^2\kappa^2}{2s(1+s)}<\rho<\frac{32}{9}\,\eta\,\frac{1+s}{s}\left[\frac{1}{4}\frac{\eta^2\kappa^2}{2s(1+s)}\right]. \tag{44}$$

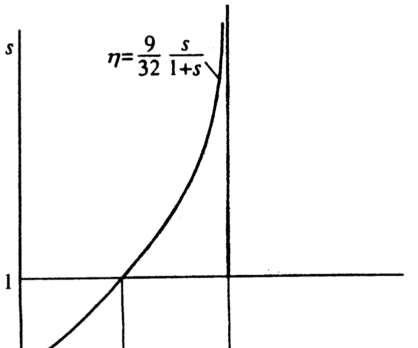

FIG. 3. Region of stability for generation of SIT pulses for a filter with $\kappa_f^2=2s(1+s)$.

Returning to our discussion of the limits of applicability of the perturbation theory we have developed, we note that the accuracy of our results depends on the extent to which the following inequalities are satisfied:

$$\tau_p \ll T_{2p} \tag{45a}$$

$$\rho \gg \rho_0 = \eta T_{2p}/\tau_p. \tag{45b}$$

The conditions (45) agree with the overall formulation of the problem. The first of them is easily satisfied due to the considerable difference between the spectral widths of the amplifier and absorber line profiles: $\kappa \gg 1$. In addition, there is no need for a special arrangement to satisfy the inequality (45b), because it is a consequence of (45a). It is not difficult to verify that when the magnitude of the pump is sufficient to satisfy (45a), the density of absorbers required for stable oscillation $\rho_{\min}$ always exceeds ρ_0.

5. DISCUSSION

We have set up the basis for a theory of mode locking with a coherent absorber, and shown that stable generation of 2π pulses is possible for a suitable choice of parameters of the amplifying and absorbing media. The profile of the envelope is shaped primarily by the absorber. The amplifier compensates the losses of the field, and consequently determines the energy and duration of the 2π pulse.

This theoretical analysis allows us to identify two factors that hinder experimental implementation of stable mode locking.

5.1. High concentration thresholds

We can propose two different methods for threshold lowering. The first is to choose an amplifying medium with a long recovery time. In this case the threshold $\rho_{\min}$ is determined not by the steady state gain, but by its saturation value. If we follow this path we must resign ourselves to losses in the generated pulse power. We should also keep in mind that the maximum achievable density of absorbers ρ_{cr} also decreases as the intraresonator gain decreases. We can increase the recovery time T_{1g} by choosing a solid-state medium as the amplifier and/or decreasing the length of the resonator.

Another method of decreasing the threshold $\rho_{\min}$ involves choosing an absorbing medium with an inhomogeneously broadened transition line. Then the absorption coefficient for a weak field in the resonator will be determined not by the phase memory time of the medium T_{2p} but by the width of the absorption line profile T_2^*. At the same time, the concentration of the coherent absorber is preserved, because coherent interactions of the field with the absorber play a decisive role when $\tau_p \ll T_{2p}$ and consequently the coherence does not depend at all on the quantity T_2^*. Once we decrease the ratio of times T_2^*/T_{2p}, we can attain stable generation of 2π pulses for moderate absorber concentrations. By decreasing ρ, we decrease the incoherent field losses at the same time, which allows us to obtain the most powerful pulses possible for a given value of the pump.

5.2. Limitations on the magnitude of the stability parameter

The need to have $s \geqslant 1$ strongly constrains our choice of the corresponding pairs of media, i.e., the laser operating medium and the absorber. Furthermore, a large value of s corresponds to a small gain saturation energy, and consequently, prevents us from obtaining the shortest possible 2π pulses. We can remove the limitation on values of s by following a path that avoids the reason for amplitude instability of the 2π pulse. We have indicated above that linear dispersion of the group velocity leads to a delay in the pulse as it propagates in the amplifier, and requires a larger value of the saturation coefficient (whose role is played by the parameter s) in order that the nonlinear correction to the group velocity be able to cancel the delay. However, there is another method for compensation that we should consider—shifting the carrier frequency of the pulse into the region of anomalous dispersion.[1)] By doing this, we obtain an acceleration of the pulse instead of a delay, and we are free of the need to limit the value of the parameter s. The carrier frequency can be shifted by moving the centers of the absorption and gain lines. A quantitative estimate of the required shift between line centers can be given only on the basis of further theoretical analysis.

The authors are unaware of work in which SIT pulses have been obtained experimentally. However, the authors of Ref. 27 mention a program for generating 2π pulses using a pumped, erbium-doped waveguide at room temperature as a gain medium, and another segment of erbium-doped waveguide at 4.2 K as a coherent absorber. We hope that the theory developed above and the proposed recommendations will be useful in subsequent experimental investigations.

[1)]The fact that the region of anomalous dispersion is located near the center of the gain line profile (at a distance $\approx T_{2g}^{-1}$) is shown, for example, in Ref. 26.

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