Quantum electromagnetic radiation of a charged particle in the field of a rotating black hole

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We consider the quantum electromagnetic radiation of a charged particle in the field of a rotating microscopic black hole. We derive transition probabilities for photon emission in various states, between quasistationary levels of a hydrogen-like "atom" with a black hole instead of a nucleus. We demonstrate that radiative transitions to low-lying energy levels in such an atom are more likely than tunneling to the black hole horizon. We show that the sum rules for the addition of angular momentum involved in photon emission break down when the black hole has nonvanishing angular momentum.

1. INTRODUCTION

Problems that relate the interaction of quantum particles with a gravitational field are of great theoretical interest. Most studies in this field have addressed quantum processes in the vicinity of microscopic black holes (mass 10^{17} g),[1] and have established that a multitude of quantum effects come into play near such a black hole, profoundly influencing its evolution. Foremost among these is the creation of particles near a collapsed hole via the Hawking process.[2] There is another process, first predicted by Zel'dovich[3] and his coworkers,[4,5] that takes place in the vicinity of a rotating black hole. It is known as *superradiant scattering*, with outgoing radiation extracting energy from the black hole.

Creation processes are localized near the horizon, where the gravitational field is more moderate (quasi-Newtonian). Here, massive particles can be captured in quantized bound states that are structurally similar to atomic states,[6,7] the difference being that these states decay via particle tunneling to the horizon. These levels are filled by particles created near the horizon that tunnel in the opposite direction.

These processes are due to interactions between a quantum field and the background gravitational field, disregarding effects of other fields. The quantum theory of interacting fields in a background gravitational field has, in turn, been studied in some detail (see e.g., Refs. 1, 8, 9); consideration has been accorded the choice of a vacuum state and the influence of particle creation on interaction processes.

In that regard, there is interest in studying the emission and absorption of quanta of the electromagnetic field by charged particles in the strong gravitational field of a microscopic black hole. In the present paper, we model such processes by examining radiation generated by charged particles in quantum bound states in the field of a black hole.

2. RADIATION OF A SCALAR PARTICLE

A system consisting of two interacting fields—a scalar field Φ with charge e and mass μ, and an electromagnetic field A^μ—can be described in curved spacetime with metric tensor $g^{\mu\nu}$ by means of the gauge invariant, generally covariant Lagrangian

$$L=(\nabla^\mu-ieA^\mu)\Phi^*(\nabla_\mu+ieA_\mu)\Phi-\mu^2\Phi^*\Phi+L_{e-m}, \tag{1}$$

where L_{e-m} is the generally covariant electromagnetic field Lagrangian. Here we assume the gravitational field to be given.

The Hamiltonian of this system of fields can be written in the form

$$H=H_{sc}+H_{e-m}+H_{\rm int}, \tag{2}$$

$$H_{\rm int}=-e\left(A^\mu J_\mu+\frac{g^{0k}}{g^{00}}A^0J_k\right)+e^2\Phi^*\Phi\left[\frac{(A^0)^2}{g^{00}}-A^\mu A_\mu\right], \tag{3}$$

$$J_\mu=i(\Phi^*\nabla_\mu\Phi-\nabla_\mu\Phi^*\Phi), \tag{4}$$

where H_{sc} and H_{e-m} are the free-scalar and electromagnetic field Hamiltonians, and $H_{\rm int}$ is the electromagnetic interaction Hamiltonian, which can be treated as a perturbation on account of the smallness of the coupling constant.

In first-order perturbation theory, the transition probability from state $|\alpha\rangle$ to state $|\beta\rangle$ is

$$W_{\alpha\beta}=2\pi|P_{\alpha\beta}|^2\delta(\omega_\alpha-\omega_\beta), \tag{5}$$

$$P_{\alpha\beta}=\langle\beta|\int_{t={\rm const}}\sqrt{g}d^3x{:}H_{\rm int}{:}|\alpha\rangle. \tag{6}$$

We consider the electromagnetic radiation of a single scalar particle in the field of a black hole, with transitions between initial and final quasistationary bound states with quantum numbers $i=\{\epsilon lm\}$ and $f=\{\epsilon' l' m'\}$, where ϵ is the particle energy, l is its orbital angular momentum, and m is its azimuthal quantum number.

The matrix element for the quantum transition of a particle emitting a photon in the field of a Kerr black hole is

$$P_{ifj}={}_{\Phi}\langle 1_f|_A\langle 1_j|\int \sqrt{g}d^3x{:}H_{\rm int}{:}|0\rangle_A|1_i\rangle_{\Phi}$$

$$=-ie\int \sqrt{g}d^3x\Bigg\{(\Phi_f^*D\Phi_i-\Phi_iD^+\Phi_f^*)A_n^*$$

$$-\frac{\Delta}{2\Sigma}(\Phi_f^*D^+\Phi_i-\Phi_iD\Phi_f^*)A_l^*$$

$$+\frac{\rho^*}{\sqrt{2}}(\Phi_f^*L_0^+\Phi_i-\Phi_iL_0\Phi_f^*)A_m^*+\frac{\rho}{\sqrt{2}}(\Phi_f^*L_0\Phi_i$$

$$-\Phi_iL_0^+\Phi_f^*)A_{m*}^*+\frac{g^{t\phi}}{g^{tt}}\Bigg[(r^2+a^2)\left(\frac{A_l^*}{2\Sigma}+\frac{A_n^*}{\Delta}\right)$$

$$-(\rho^*A_m^*-\rho A_{m*}^*)\Phi_f^*L_0\Phi_i(m+m')\frac{a}{\sqrt{2}}\sin\theta\Bigg]\Bigg\}. \tag{7}$$

Here we employ the notation of Newman and Penrose[10] and Press and Teukolsky[11]:

$$\Sigma=r^2+a^2\cos^2\theta,\quad \Delta=r^2-2Mr+a^2,$$

$$\rho=-(r-ia\cos\theta)^{-1},$$

$$D=\frac{\partial}{\partial r}-\frac{i}{\Delta}[\omega(r^2+a^2)-\nu a],\quad D^+=D(-\omega,-\nu),$$

$$L_s=\frac{\partial}{\partial\theta}+\frac{\nu}{\sin\theta}-a\omega\sin\theta+s\cot\theta,$$

$$L_s^+=L(-\omega,-\nu), \tag{8}$$

where M is the mass of the black hole, a is its angular momentum per unit mass, t, r, θ, and ϕ are spherical coordinates, and A_m, A_l, and A_n are the components of the electromagnetic vector potential in the Kinnersley basis:

$$A^\mu=A_ln^\mu+A_nl^\mu-A_mm^{*\mu}-A_{m*}m^\mu,$$

$$n^\mu=(2\Sigma)^{-1}[r^2+a^2,-\Delta,0,a],$$

$$l^\mu=\Delta^{-1}[r^2+a^2,\Delta,0,a],$$

$$m^\mu=-\frac{\rho^*}{\sqrt{2}}\left[ia\sin\theta,0,1,\frac{i}{\sin\theta}\right].$$

We assume that the operators L and D act on wave functions whose dependence upon t and ϕ takes the form $\exp(-i\omega t+i\nu\phi)$.

Ternov *et al.*[6] derived a bound-state particle wave function defined both with the localization region of the particle and outside it, near the horizon. Its frequency contains an imaginary correction $-i\delta$, which acts as a wave function damping constant. For present purposes, a separate description of the two quantum states, with real frequencies, is more reasonable: one in the particle localization region, and the other outside it, near the horizon. Transitions between the states are described by the transition Hamiltonian $H_{\rm tran}=\delta(b^+c+c^+b)$, where c, c^+ and b, b^+ are particle annihilation and creation operators in the localized state and near the horizon, respectively.

Given that $\mu M\ll 1$, the wave function for the bound state of a scalar particle in the localization region $r\sim a_0\gg r_+$ ($a_0=(\mu^2M)^{-1}$ is the typical size of an "atom," r_+ is the radius of the black hole horizon, and $r_\pm=M\pm\sqrt{M^2-a^2}$) is then

$$\Phi(\epsilon ml,x)=\frac{\exp(-i\epsilon t+im\phi)}{\sqrt{2\pi}\sqrt{2\epsilon}}\cdot{}_0R_{lm}^{\epsilon}(r)_0S_{lm}(a,\theta). \tag{9}$$

The radial function ${}_0R_{lm}^{\epsilon}(r)$ can be approximated in terms of the degenerate hypergeometric function $\Phi(1+l-n,\ 2l+2,q)$, where n is the principal quantum number of the particle's energy level in a Newtonian potential[6] when the particle energy is ϵ:

$${}_0R_{lm}^{\epsilon}(r)\approx R_{nl}(r)=C_{nl}\exp(-q/2)q^l\Phi(1+l-n,2l+2,q), \tag{10}$$

$$C_{nl}=\sqrt{\frac{4\ (n+1)!}{n(n-l-1)!}}\frac{1}{(2l+1)!}\frac{1}{(na_0)^{3/2}},$$

$$q=\frac{2r}{a_0n},\quad \epsilon=\mu\left[1-\frac{1}{2}\left(\frac{\mu M}{n}\right)\right]. \tag{11}$$

In Eq. (9), ${}_0S_{lm}(a,\theta)$ is a spheroidal wave function, which can be expanded in powers of $a^2(\mu^2-\epsilon^2)$ with the help of the associated Legendre polynomials P_l^m:

$${}_0S_{lm}(a,\theta)=\sqrt{\left(l+\frac{1}{2}\right)\frac{(l+m)!}{(1-m)!}}P_l^m(\cos\theta)-a^2(\mu^2-\epsilon^2)$$

$$\times[\zeta P_{l+2}^m(\cos\theta)+\xi P_{l-2}^m(\cos\theta)]$$

$$+o[a^2(\mu^2-\epsilon^2)],$$

$$\zeta=\sqrt{\frac{[(l+1)^2-m^2][(l+2)^2-m^2]}{2l+1}}$$

$$\times\sqrt{\frac{(l+2+m)!}{(l+2-m)!}}\frac{1}{2\sqrt{2}(2l+3)^2},$$

$$\xi=\sqrt{\frac{[(l-1)^2-m^2](l^2-m^2)}{2l+1}}$$

$$\times\sqrt{\frac{(l-2+m)!}{(l-2-m)!}}\frac{1}{2\sqrt{2}(2l-1)^2}. \tag{12}$$

The state of a photon in the field of a black hole can be characterized by its energy ω, azimuthal quantum number ν, dipole moment J, and polarization P, plus the asymptotic behavior of the wave function at the black hole horizon H^- and at null infinity J^-. We can write an expression for the *in* mode,[12] corresponding to a particle arriving from J^- (with $r=\infty$) in the gauge $A_l=0$, that is compatible with the normalization criterion at J^- (Ref. 13):

$$A_{\rm in}^\mu(x,\omega\nu JP=\pm1)=(1\pm\Pi)\left[\frac{l^\mu\rho}{\sqrt{2}}(L_1-ia\rho^*\sin\theta)\right.$$

$$\left.+m^{*\mu}(D+\rho^*)\right]$$

TABLE I. Transition probabilities for a scalar particle with photon emission in an in mode.

Δl	$W_{in}(nlm \to n'l'm')$
-1	$\frac{4}{3}e^2\omega^3\left[\frac{C_{n',l-1}C_{nl}}{n'^{l-1}n^l}\left(\frac{a_0}{2}\right)^4 J^{(1,2)}_{2l+1}(n-l-1,n'-l)Y^{l-1,m-\nu}_{l,m}\right]^2$
-3	$\frac{4}{3}e^2\omega^3\left[\frac{C_{n',l-3}C_{nl}}{n'^{l-3}n^l}\left(\frac{a_0}{2}\right)^4 J^{(-1,6)}_{2l+1}(n-l-1,n'-l+2)Y^{l-3,m-\nu}_{l,m}\right]^2$
0	$\frac{4}{3}e^2\omega^3\left[\frac{C_{n',l}C_{nl}}{n'^{l}n^l}\frac{\omega}{\mu}\left(\frac{a_0}{2}\right)^4 J^{(2,0)}_{2l+1}(n-l-1,n'-l-1)Y^{l,m-\nu}_{l,m}\right]^2$
-2	$\frac{4}{3}e^2\omega^3\left[\frac{C_{n',l-2}C_{nl}}{n'^{l-2}n^l}a\left(\frac{a_0}{2}\right)^3 J^{(-1,4)}_{2l+1}(n-l-1,n'-l+1)Y^{l-2,m-\nu}_{l,m}\right]^2$

$$\times {}_{-1}R^{\rm in}_{J\omega}(r)\,{}_1S^{\omega}_{Jn}(a,\theta)\frac{\exp(-i\omega t+i\nu\phi)}{\sqrt{2\pi}}. \qquad (13)$$

Here, Π is the spatial reflection operator: $\Pi\varphi(t,r,\theta,\phi)=\varphi(t,r,-\theta,-\phi)$.

We can write an expression in the gauge $A_n=0$ for the *up* mode corresponding to a particle departing from H^- that is compatible with the normalization condition at H^-:

$$A^{\mu}_{\rm up}(x,\omega\nu JP=\pm 1)=(1\pm\Pi)\left[\frac{n^\mu}{\sqrt{2}}\left(-\frac{L_1^+}{\rho^*}+ia\sin\theta\right)+\frac{m^\mu\Delta}{2\Sigma\rho^{*2}}\left(D^2+\rho^*+\frac{2(r-M)}{\Delta}\right)\right]$$
$$\times {}_1R^{\rm up}_{J\omega}(r)\,{}_{-1}S^{\omega}_{Jn}(a,\theta)\frac{\exp(-i\omega t+i\nu\phi)}{\sqrt{2\pi}}. \qquad (14)$$

From here on, we consider only dipole radiation ($J=1$). In the localization region $r\sim a_0$ of a scalar particle, the radial function ${}_{\pm 1}R_{J\omega}(r)$ can be expanded in series:

$$ {}_{-1}R^{\rm in}_{J\omega}(r)\approx\frac{1}{\sqrt{4\pi\omega}}\left(\frac{4}{3}\omega^2r^2+\frac{2}{3}i\omega^3r^3+\dots\right), \qquad (15)$$

$$ {}_1R^{\rm up}_{J\omega}(r)\approx\sqrt{\frac{\tilde\omega}{8\pi Mr_+}}\frac{2Mr_+(r_+-r_-)}{\sqrt{1+[4Mr_+\tilde\omega/(r_+-r_-)^2}}\frac{B}{r^3}, \qquad (16)$$

where $\tilde\omega=\omega-\nu\Omega,\Omega=a(2Mr_+)^{-1}$ is the rotation rate of the black hole, and the coefficient B, which can be determined by the method given in Ref. 5, is equal to 1/3.

The spheroidal wave functions ${}_sS^{\omega}_{Jn}(a,\theta)$ can be expressed in terms of orthogonal polynomials $P^J_{n,-s}$ given in Ref. 14:

$$ {}_sS^{\omega}_{Jn}(a,\theta)=\sqrt{\frac{2J+1}{2}}\{P^J_{n,-s}(\cos\theta)+a\omega[\sigma_{n+1}P^J_{n+1,-s}(\cos\theta)+\sigma_{n-1}P^J_{n-1,-s}(\cos\theta)]\},$$
$$\sigma_1=1/4,\quad \sigma_2=1/4,\quad \sigma_0=-1/4. \qquad (17)$$

The *in* and *up* modes form a basis in the state space, in which it is necessary to define the quantization method and the vacuum state. The case of a permanent black hole corresponds to η-quantization, in which modes with $\tilde\omega>0$ are deemed to be positive-frequency[15]:

TABLE II. Angular integrals contributing to transition probabilities for a scalar particle.

Δl	$Y^{l',m'}_{l,m}$
-1	$\sqrt{\frac{l}{2l+1}}\langle l-1\;1\;m-\nu\;\nu\mid l\;m\rangle$
-3	$(a\mu)^2(\mu M)^2\left[\frac{1}{[(2l-1)n]^2}+\frac{1}{[(2l-3)n']^2}\right]\sqrt{\frac{l-2}{2l+1}}\times$ $\times\frac{\sqrt{(l^2-m^2)[(l-1)^2-m^2]}}{2(2l-1)(2l-3)}\langle l-3\;1\;m-\nu\;\nu\mid l-2\;m\rangle$
0	$\sqrt{l(l+1)}\langle l\;1\;m-\nu\;\nu\mid l\;m\rangle$
-2	$\frac{\sqrt{l^2-m^2}}{2(2l-1)}\sqrt{\frac{l-1}{2l-3}}\langle l-2\;1\;m-\nu\;\nu\mid l-1\;m\rangle$

$$A^{\mu}=\int_0^{\infty}d\omega\sum_{\nu JP}(c_{\rm in}A^{\mu}_{\rm in}+c^+_{\rm in}A^{\mu *}_{\rm in})$$

$$+\int_0^{\infty}d\tilde{\omega}\sum_{\nu JP}(c^{\eta}_{\rm up}A^{\mu}_{\rm up}+c^{\eta +}_{\rm up}A^{\mu *}_{\rm up}), \quad (18)$$

where c and c^+ are the photon annihilation and creation operators.

Unruh proposed[16] ξ-quantization as a means of describing the state of the electromagnetic field (which is characterized by a flux of Hawking particles) following gravitational collapse. The *up* modes are then positive-frequency for all $\tilde{\omega}$, and an additional factor is incorporated beyond Eq. (14):

$$A^{\mu}=\int_0^{\infty}d\omega\sum_{\nu JP}(c_{\rm in}A^{\mu}_{\rm in}+c^+_{\rm in}A^{\mu *}_{\rm in})$$

$$+\int_{-\infty}^{\infty}d\tilde{\omega}\sum_{\nu JP}(c^{\xi}_{\rm up}pA^{\mu}_{\rm up}+c^{\xi +}_{\rm up}pA^{\mu *}_{\rm up}),$$

$$p(\tilde{\omega})=\exp\left(\frac{\pi\tilde{\omega}}{2k}\right)\frac{1}{\sqrt{2\sinh(\pi|\tilde{\omega}|/k)}},\quad k=\frac{r_+-r_-}{4Mr_+}. \quad (19)$$

The initial state of the electromagnetic field is taken to be the vacuum, defined by $c|0\rangle_A=0$ for all operators c: $c_{\rm in}$, $c^{\eta}_{\rm up}$ for η-quantization (Boulware vacuum), and $c_{\rm in}$, $c^{\xi}_{\rm up}$ for ξ-quantization (Unruh vacuum).

Substituting (8)–(19) into the matrix element (7), we find the following selection rules for electromagnetic radiation

1) energy: $\epsilon'=\epsilon-\omega$,

2) azimuthal quantum number: $m'=m-\nu$, (20)

3) parity: $P=(-1)^{l-l'}$.

If the metric is spherically symmetric (like the Schwarzschild metric, for example), then

$$|l'-l|<J<l'+l, \quad (21)$$

which is analogous to the rule for the addition of angular momentum in the hydrogen atom. The Kerr metric, on the other hand, is not spherically symmetric, and the rule for adding angular momenta does not hold.

Table I lists expressions for several transition probabilities between bound states involving the emission of a TM photon in an *in* mode with $J=1$ if Δl is odd, and a TE photon if Δl is even.

The quantity $J^{(\sigma,\tau)}_{\rho}(p,p')$ in the expression for the probability takes the form

$$J^{(\sigma,\tau)}_{\rho}(p,p')=\int_0^{\infty}d\xi\xi^{\rho+\sigma}\exp\left[-\frac{\xi}{2}\left(\frac{1}{n}+\frac{1}{n'}\right)\right]$$

$$\times\Phi\left(-p,\rho+1,\frac{\xi}{n}\right)\Phi\left(-p',\rho+1-\tau,\frac{\xi}{n'}\right)$$

$$=\sum_{\nu=0}^{p}\sum_{\nu=0}^{p'}\frac{p!p'!\rho!(\rho-\tau)!(-1)^{\mu+\nu}(\nu+\mu+\rho+\sigma)!}{(p-\nu)!(p'-\nu)!(\rho+\nu)!(\rho-\tau+\mu)!\nu!\mu!n^{\nu}n'^{\mu}}\left(\frac{2nn'}{n+n'}\right)^{\nu+\mu+\rho+\sigma+1},$$

and values of the $Y^{l,m-\nu}_{l,m}$ are given in Table II in terms of Clebsch–Gordan coefficients.

In either type of vacuum, radiative transitions only take place with particle energy losses and photon creation in an *in* state ($\epsilon'<\epsilon$, $\omega>0$).

According to the rule for the addition of angular momenta, only transitions with $\Delta l=0,\pm 1$ are permitted in dipole radiation. The transition probabilities, which are subject to the addition rule, are the same in the "particle–black hole" system and the hydrogen atom. There is no discernible dependence of the probability of such transitions on the black hole rotation rate.

The transition probabilities in Table I have the following orders of magnitude:

$$W(\Delta l=\pm 1)\sim e^2(\mu M)^4\mu,$$

$$W(\Delta l=\pm 2)\sim e^2(\mu a)^2(\mu M)^6\mu,$$

$$W(\Delta l=0)\sim e^2(\mu M)^8\mu,$$

$$W(\Delta l=\pm 3)\sim e^2(\mu a)^4(\mu M)^8\mu. \quad (22)$$

Clearly, even for a rotating black hole, for which the angular momentum addition rule does not strictly hold, transition probabilities that violate the addition rule tend to vanish in the limit as Δl increases. The only exception is the case in which $a\approx M$, for which the probabilities of magnetic transitions with $\Delta l=0$ and $\Delta l=\pm 2$ are of comparable magnitude.

The probabilities of the most intense transitions with photon emission in an *up* state are shown in Table III; depending on the type of vacuum—Unruh or Boulware—they differ by a factor $p^2(\tilde{\omega})$:

$$W_U(nlm\to n'l'm')=p^2(\tilde{\omega})W_B(nlm\to n'l'm'). \quad (23)$$

This results in large differences in order of magnitude:

$$W_B(\Delta l=\pm 1)\sim e^2(\mu M)^{10}\mu,$$

$$W_U(\Delta l=\pm 1)\sim e^2(\mu M)^{7}\mu,$$

$$W_B(\Delta l=0)\sim e^2(\mu M)^{12}\mu,$$

$$W_U(\Delta l=0)\sim e^2(\mu M)^{9}\mu. \quad (24)$$

TABLE III. Transition probabilities for a scalar particle with photon emission in an *up* mode.

Δl	$W_{up}(nlm \to n'l'm')$
-1	$\frac{4}{3}e^2\tilde{\omega}\frac{Mr_+}{2}(r_+ - r_-)^2 \times$ $\times\left[\frac{C_{n',l-1}C_{nl}}{n'^{l-1}n^l}\frac{a_0}{2}J_{2l+1}^{(-2,2)}(n-l-1,n'-l)Y_{l,m}^{l-1,m-\nu}\right]^2$
0	$\frac{4}{3}e^2\tilde{\omega}\frac{Mr_+}{2}(r_+ - r_-)^2\left[\frac{C_{n',l}C_{nl}}{n'^l n^l}\frac{1}{\mu}J_{2l+1}^{(-2,0)}(n-l-1,n'-l-1)Y_{l,m}^{l,m-\nu}\right]^2$

In the vicinity of a black hole in the Boulware vacuum, transitions involving the emission of an *up* photon with energy $\tilde{\omega}>0$ are permitted. When $\tilde{\omega}>-\nu\Omega$, the particle loses energy: $\epsilon'=\epsilon-\tilde{\omega}-\nu\Omega<\epsilon_i$. Conversely, upon emission of a photon with $\tilde{\omega}<-\nu\Omega$, it gains energy: $\epsilon'>\epsilon$. The absorption of energy from the vacuum is associated with the nonvanishing energy-momentum tensor in the vacuum state of a rotating black hole.

Additionally, transitions involving photon creation with energy $\tilde{\omega}<0$ are permitted in the Unruh vacuum, so the possibility that a particle will extract energy from the vacuum is significantly enhanced.

Nevertheless, it can easily be seen by comparing (22) and (24) that photon emission in *in* states is much more likely than emission in *up* states. The most likely transition with a photon in an *up* mode in the Unruh vacuum is $(\mu M)^{-3}\gg 1$ times less probable than the most likely transition with photon creation in an *in* mode.

The radiative transition probabilities given in Tables I and III govern the interaction between a scalar particle in a bound state in the field of a black hole and the electromagnetic vacuum.

3. RADIATION OF A SPINOR PARTICLE

The Hamiltonian for the interaction of a spinor field with an electromagnetic field takes the form

$$H_{\text{int}}=ie\bar{\psi}\gamma^{\mu}\psi A_{\mu}, \tag{25}$$

where the $\gamma^{\mu}(x)$ are gamma matrices in curved spacetime. We consider only one-particle states. The matrix element for a transition between such states that is accompanied by the creation of a photon in the electromagnetic vacuum can be written in the following manner, using Newman–Penrose notation:

$$\begin{aligned}P_{ifj}&={}_{\Psi}\langle 1_f|_A\langle 1_j|\int\sqrt{g}d^3x{:}H_{\text{int}}{:}|0\rangle_A|1_i\rangle_{\Psi}\\&=\sqrt{2}\int\sqrt{g}\,\frac{d^3x}{2\pi}\exp[i(m-m')\phi]\\&\quad\times(S_f^{-*}S_i^{-}+S_f^{+*}S_i^{+})\left(\frac{R_f^{+*}R_i^{+}A_n^*}{\Delta}+\frac{R_f^{-*}R_i^{-}A_l^*}{2\Sigma}\right)\\&\quad+(\rho S_f^{+*}S_i^{-}A_m^*-\rho^*S_f^{-*}S_i^{+}A_{m*}^*)\frac{R_f^{+*}R_i^{+}-R_f^{-*}R_i^{+}}{2\Delta}.\end{aligned}\tag{26}$$

Here we represent the spinor wave function in a form derived by Chandrasekhar[17]:

$$\psi(\epsilon mjs,x)=\frac{\exp(-i\epsilon t+im\phi)}{\sqrt{2\pi}}\left(-\frac{\rho S^{-}(\theta)R^{-}(r)}{\sqrt{2}},\frac{S^{+}(\theta)R^{+}(r)}{\sqrt{\Delta}},-\frac{S^{-}(\theta)R^{+}(r)}{\sqrt{\Delta}},\frac{\rho^*S^{+}(\theta)R^{-}(r)}{\sqrt{2}}\right)^T.\tag{27}$$

The spheroidal wave functions $S(\theta)$ in (26) can be expanded in orthogonal polynomials[14]:

$$\begin{aligned}S_{jm}^{-}(s,\theta)&=\sqrt{\frac{1}{2}\left(j+\frac{1}{2}\right)}[P_{m,1/2}^{j}(\cos\theta)+a(\epsilon+s\mu)\kappa_{j+1}\\&\quad\times P_{m,1/2}^{j+1}(\cos\theta)-a(\epsilon-s\mu)\kappa_jP_{m,1/2}^{j-1}(\cos\theta)]\\&\quad+O[a^2(\mu^2+\epsilon^2)],\end{aligned}$$

$$\kappa_j=\frac{\sqrt{j^2-m^2}}{(2j)^2},\quad S_{j,m}^{+}(s,\theta)=-s(-1)^{j-m}S_{j,m}^{-}(s,\pi-\theta),\tag{28}$$

where $s=\pm 1$, j and m are half-odd integers, and $|m|\leq j$.

With $\mu M\ll 1$ at $r\gg r_+$, the radial wave functions of the quasistationary state take the form[18,19]

TABLE IV. Transition probabilities for a spinor particle with photon emission in an *in* mode.

Δl	Δj	$W_{in}(nljm \to n'l'j'm')$
-1	0, -1, -2	$\frac{4}{3}e^2\omega^3\left[\frac{C_{n',l-1}C_{nl}}{n'^{l-1}n^l}\left(\frac{a_0}{2}\right)^4 J_{2l+1}^{(1,2)}(n-l-1,n'-l)Y_{j,m,s}^{j',m-\nu,s'}\right]^2$
-3	-2, -3	$\frac{4}{3}e^2\omega^3\left[\frac{C_{n',l-3}C_{nl}}{n'^{l-3}n^l}\left(\frac{a_0}{2}\right)^4 \times J_{2l+1}^{(-1,6)}(n-l-1,n'-l+2)Y_{j,m,s}^{j-3,m-\nu,s'}\right]^2$
0	0, -1	$\frac{4}{3}e^2\omega^3\left[\frac{C_{n',l}C_{nl}}{n'^l n^l}\frac{\omega}{\mu}\left(\frac{a_0}{2}\right)^4 J_{2l+1}^{(2,0)}(n-l-1,n'-l-1) \times\frac{1}{2}\left(s\left(j+\frac{1}{2}\right)+s'\left(j'+\frac{1}{2}\right)\right)Y_{j,m,s}^{j',m-\nu,s'}\right]^2$
-2	-1	$\frac{4}{3}e^2\omega^3\left(\frac{C_{n',l-2}C_{nl}}{n'^{l-2}n^l}\left(\frac{a_0}{2}\right)^4\right)^2 \times\left[\left(\frac{\omega}{2\mu}J_{2l+1}^{(0,4)}(n-l-1,n'-l+1)Y_{j,m,s}^{j-1,m-\nu,-s}\right)^2 +\left(\frac{2a}{a_0}J_{2l+1}^{(-1,4)}(n-l-1,n'-l+1)\tilde{Y}_{j,m,s}^{j-1,m-\nu,-s}\right)^2\right]$
-2	-2	$\frac{4}{3}e^2\omega^3\left[\frac{C_{n',l-2}C_{nl}}{n'^{l-2}n^l}a\left(\frac{a_0}{2}\right)^3 \times J_{2l+1}^{(-1,4)}(n-l-1,n'-l+1)Y_{j,m,s}^{j-2,m-\nu,s}\right]^2$

$$\operatorname{Im} R^-(r)\approx 2^{-1/4}C_{nl}\exp(-q/2)rq^l\Phi(1+l-n,2l+2,q),$$

$$\operatorname{Re} R^-(r)=\frac{1}{2\mu r}\left[s\left(j+\frac{1}{2}\right)+r\frac{d}{dr}\right]\operatorname{Im} R^-(r),$$

$$R^+(r)=R^{-*}(r), \tag{29}$$

where C_{nl} and q are given by (11), and $l=j+s/2$.

Substituting (27)–(29) and the photon wave functions (13)–(16) with quantum numbers $\{\omega,P,J,\nu\}$ into (26), we obtain exactly the same energy, azimuthal quantum number, and parity selection rules (20) as for a scalar particle.

The rule for the addition of angular momenta for a spinor particle,

$$|j'-j|<J<j'+j, \tag{30}$$

where j and J are respectively the total momentum of particle and photon, holds in the field of a Schwarzschild black hole, and breaks down in the field of a rotating black hole.

Table IV gives probability expressions $W_{in}(njlm \to n'j'l'm')$ for some of the most intense transitions between bound states involving photon emission in an *in* state with $J=1$ (with electric or dipole radiation). Values of $Y_{jms}^{j'm's'}$ are given in Table V.

The expressions for the probability of normal transitions with $\Delta l=0,\pm 1,\pm 2$ (which satisfy the angular momentum addition rule) are the same in the field of a Schwarzschild black hole and for a hydrogen atom. In transitions with no spin flip ($\Delta l=0,\pm 2$), the dependence of the transition probability on the black hole rotation rate is negligible, while the probability of normal spin-flip transitions ($\Delta l=\pm 1,\Delta j=0$) depends heavily on the rotation rate.

Table IV gives the order of magnitude of the transition probabilities

$$W(\Delta l=\pm 1,\Delta j=\pm 1)\sim W(\Delta l=\pm 1,\Delta j=0)\sim e^2(\mu M)^4\mu,$$
$$W(\Delta l=0)\sim W(\Delta l=\pm 2,\Delta j=\pm 1)\sim e^2(\mu M)^8\mu,$$
$$W(\Delta l=\pm 1,\Delta j=\pm 2)\sim e^2(\mu a)^2(\mu M)^4\mu,$$
$$W(\Delta l=\pm 3,\Delta j=\pm 2)\sim e^2(\mu a)^2(\mu M)^8\mu,$$
$$W(\Delta l=\Delta j=\pm 2)\sim e^2(\mu a)^2(\mu M)^6\mu,$$
$$W(\Delta l=\Delta j=\pm 3)\sim e^2(\mu a)^4(\mu M)^8\mu. \tag{31}$$

Probabilities of anomalous transitions, which violate the angular momentum addition rule, fall off with increasing Δl, and when $\mu a\ll 1$ and $\mu M\ll 1$, they are much lower than probabilities of normal transitions, which conform to the rule. The probabilities of transitions with $\Delta l=\Delta j$ are of the same order of magnitude as for a scalar particle with the same Δl and Δm (for either type of transition). Note that

TABLE V. Angular integrals contributing to transition probabilities for a spinor particle.

Δj	ss'		$Y_{j,m,s}^{j',m',s'}$
-1	1		$\frac{1}{2}\sqrt{\frac{2j-1}{j}}\langle j-1\,1\,m-\nu\,\nu \mid j\,m\rangle$
-1	-1	Y	$\frac{1}{2}\sqrt{\frac{2j-1}{j}}\langle j-1\,1\,m-\nu\,\nu \mid j\,m\rangle$
		$\tilde{Y}$	$\frac{1}{\sqrt{(4j)^3}}\left(m\frac{\sqrt{2j-1}}{j+1}\langle j-1\,1\,m-\nu\,\nu \mid j\,m\rangle + \frac{\sqrt{(j^2-m^2)(j+1)}}{j-1}\langle j\,1\,m-\nu\,\nu \mid j\,m\rangle\right)$
-3	1		$a^2(\epsilon-s\mu)(\epsilon'+s\mu)\sqrt{\frac{(j^2-m^2)((j-1)^2-m^2)(2j+1)}{(2j-3)(2j-1)}} \times\frac{2j-5}{8(j-1)\sqrt{(j-2)^3}}\frac{1}{4j^2}\langle j-3\,1\,m-\nu\,\nu \mid j-2\,m\rangle$
0	-1		$\frac{1}{2}\sqrt{\frac{j+1}{j}}\langle j\,1\,m-\nu\,\nu \mid j\,m\rangle$
0	1		$\frac{j+1/2}{\sqrt{j(j+1)}}\langle j\,1\,m-\nu\,\nu \mid j\,m\rangle$
-2	-1		$\left[a(\epsilon'-s\mu)\frac{\sqrt{j-3/2}}{j-1} - a(\epsilon-s\mu)\frac{\sqrt{j+1/2}}{j}\right] \times\frac{1}{8j}\sqrt{\frac{j^2-m^2}{j-1}}\langle j-2\,1\,m-\nu\,\nu \mid j-1\,m\rangle$
-2	1		$\frac{1}{8j}\sqrt{\frac{(j^2-m^2)(2j-3)}{j-1}}\left(1+\sqrt{\frac{j-1/2-s}{j-1/2}}\right) \times\langle j-2\,1\,m-\nu\,\nu \mid j-1\,m\rangle$

spin-flip transitions with $\Delta l=\pm 1$, $\Delta j=\pm 2$ are the most frequent of the anomalous transitions, and for $\mu a>(\mu M)^2$, they are even more likely that normal magnetic transitions with $\Delta l=0$.

The differences between the selection rules in a black hole+particle system and a hydrogen atom are due to distortion of the angular part of the particle and photon wave functions resulting from black hole rotation. The rotation of a gravitational field exerts a major influence on a multicomponent field, so in the case of a scalar particle, rotational gravitational corrections to the electromagnetic field of order $O(a\omega)$ engender magnetic dipole transitions with $\Delta l=\pm 2$ with higher probability than electric dipole transitions with $\Delta l=\pm 3$, which are induced by corrections to the wave function of the scalar particle. The $O(a\mu)$ correction to the spinor particle wave function engenders even more likely $\Delta l=\pm 1$, $\Delta j=\pm 2$ spin-flip electric dipole transitions.

Table VI lists the probabilities of the most intense transitions between quasistationary levels in which a photon with $J=1$ is emitted in the *up* state.

Like a scalar particle emitting a photon in the *up* state in the Boulware vacuum, a spinor particle goes to a higher energy level if the photon satisfies the superradiant scattering condition $\tilde{\omega}<-\nu\Omega(\tilde{\omega}>\omega-\nu\Omega>0)$.

In the Unruh vacuum, the photon emission probability in an *up* state is given by Eqs. (23) and (19); photon emission in a state with $\tilde{\omega}>-\nu\Omega$ is accompanied by a decrease in particle energy, while the energy increases in a state with $\tilde{\omega}<-\nu\Omega$.

The orders of magnitude of the transition probabilities involving photon creation in the Boulware and Unruh vacuums are

$$W_B(\Delta l=\pm 1)\sim e^2(\mu M)^{10}\mu,$$

$$W_U(\Delta l=\pm 1)\sim e^2(\mu M)^7\mu,$$

TABLE VI. Transition probabilities for a spinor particle with photon emission in an *up* mode.

Δl	Δj	$W_{up}(nljm \to n'l'j'm')$
-1	0, -1	$\frac{4}{3}e^2\tilde{\omega}\frac{Mr_+}{2}(r_+ - r_-)^2 \times \left[\frac{C_{n',l-1}C_{nl}}{n'^{l-1}n^l}\frac{a_0}{2}J_{2l+1}^{(-2,2)}(n-l-1,n'-l)Y_{j,m,s}^{j',m-\nu,s'}\right]^2$
0	0, -1	$\frac{4}{3}e^2\tilde{\omega}\frac{Mr_+}{2}(r_+ - r_-)^2 \times \left\{\frac{C_{n',l}C_{nl}}{n'^{l}n^l}\frac{1}{\mu}J_{2l+1}^{(\Delta l-2,-2\Delta l)}(n-l-1,n'-l-1) \times \frac{1}{2}\left[s'\left(j'+\frac{1}{2}\right)+s\left(j+\frac{1}{2}\right)+2\right]Y_{j,m,s}^{j',m-\nu,s'}\right\}^2$
-2	-1	

$$W_B(\Delta l=0,\pm 2)\sim e^2(\mu M)^{12}\mu,$$
$$W_U(\Delta l=0,\pm 2)\sim e^2(\mu M)^{9}\mu. \qquad (32)$$

The probability of photon emission in an *in* state is much higher than in an *up* state.

The emission of photons in an *up* state by a charged particle with spin is thus little different from emission by a spinless particle. Our analysis of the appropriate orders of magnitude suggests than in either case, processes in which electromagnetic particles are created near the black hole horizon have little influence on the evolution of particles in bound states with $\mu M \ll 1$.

It is interesting to compare the rates of two competing processes, namely the tunneling of particles out of a bound hydrogen-like level to the black hole horizon, and the fall of a particle to the lowest-lying energy levels with emission of a photon. It is well known that the tunneling probability falls off rapidly with increasing orbital angular momentum l and principal (energy) quantum number n.[6,18,19] An estimate shows that if a particle with the mass of the electron is captured to a quasistationary level by the field of a microscopic black hole of mass $M<10^{15}$g, a particle transition to the lowest-lying energy level with photon emission is more likely that decay of the level via tunneling to the horizon for all states of nonvanishing orbital angular momentum; for $l=0$, the rates of these two processes are comparable in order of magnitude. Thus, when a particle is captured by a black hole, it will first shed its energy and angular momentum via radiation, and tunneling will take place principally from an s state—primarily from the lowest-lying hydrogen-like $1s$ state.

The inverse of tunneling to the horizon is the filling of bound states with Hawking particles. The nonzero mass of these particles means that this process proceeds more slowly[2] than Hawking radiation of massless photons, and therefore the advent of another (Hawking) particle in a bound state during one particle lifetime is highly unlikely. The filling of quasistationary states with Hawking particles and the evolution of particles in these states via radiation and tunneling back out to the black hole horizon must therefore be viewed as sequential, nonsimultaneous processes.

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