Solitons in quasi-one-dimensional noncollinear antiferromagnets

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(Submitted 13 November 1995)
Zh. Éksp. Teor. Fiz. **110**, 584–599 (August 1996)

We use the method of effective Lagrangians to study the nonlinear dynamics of the simple model of a quasi-one-dimensional multiple-sublattice antiferromagnet. We analyze the structure of dynamic and topological solitons and calculate the constants of motion. Finally, we find the soliton density in thermodynamic equilibrium and calculate the dynamic structural factors generated by solitons. © *1996 American Institute of Physics.* [S1063-7761(96)01608-3]

1. INTRODUCTION

Recently there has been steady interest in theoretical and experimental investigations of various physical properties of systems close in their magnetic properties to one-dimensional systems. The interest in low-dimensional objects can be explained by the fact that the physical properties of such systems differ dramatically in some respects from the corresponding properties of three-dimensional magnetic materials. One of the most interesting differences between quasi-one-dimensional (1D) or quasi-two-dimensional systems and three-dimensional systems is that in describing their low-temperature thermodynamic properties we must allow not only for linear excitations (magnons) but also for essentially nonlinear excitations: magnetic solitons, kinks (domain walls), and vortices. Although the density of nonlinear excitations at low temperatures is low compared to the magnon density, occasionally the contribution of these excitations to the thermodynamic characteristics can dominate.

Nonlinear excitations and their contribution to the thermodynamic characteristics of one-dimensional systems have now been investigated for a number of scalar models of magnetic materials (of the sine-Gordon type; see, e.g., Izyumov's review[1]), ferromagnets described by a three-dimensional unit vector, and collinear antiferromagnets described by the σ-model.[2–4] Soliton solutions have also been studied for more complicated magnetic structures: amorphous magnetic material of the spin glass type,[5–7] multiple-sublattice magnetic materials with a modulated magnetic structure of the $CsCuCl_3$ type,[8] the four-sublattice antiferromagnet La_2CuO_4 (see Ref. 9), and magnetic materials with triangular spin structure.[10] Soliton states in noncollinear magnetic materials differ dramatically from solitons in collinear magnetic materials: their description is based on dynamical equations for variables of a different type (say, a four-dimensional unit vector; see below), they differ in their topological classification, etc.

Although noncollinear magnetic materials can be low-dimensional (e.g., the $CsCuCl_3$ compound mentioned earlier, magnetic materials with a triangular structure, and some molybdates of rare-earth and transition metals with a chain-like spin structure[11]), the thermodynamics of solitons in such systems, to our knowledge, has yet to be studied.

In this paper we use a simple model to examine solitons and the specifically soliton contribution to the thermodynamic characteristics in a one-dimensional model system with a noncollinear spin structure.

2. THE MODEL

In analyzing the dynamical properties of a multiple-sublattice magnetic material we employ a phenomenological approach. Accordingly, the long-wave spin dynamics is described in terms of the angle of rotation of a "rigid" spin basis determining the equilibrium directions of the sublattices of the noncollinear magnetic material.[12] The dynamical variable here is the orthogonal rotation matrix $\mathbf{R}_{ik}(\mathbf{r},t)$, which is specified at each point of the magnetic material and describes the rotation of the spins of the nth sublattice $\mathbf{S}_n$ from its equilibrium direction $\mathbf{S}_n^{(0)}$

$$S_{n,i}(\mathbf{r},t)=R_{ij}(\mathbf{r},t)S_{n,j}^{(0)}. \tag{1}$$

Following the general ideas of the method of effective Lagrangians[12–14] as applied to ordered media, we describe the dynamics of a system by a vector field $\boldsymbol{\varphi}$ determining a parametrization of the SO(3) group. The system Lagrangian is then expressed in terms of differential Cartan forms $\boldsymbol{\omega}(\boldsymbol{\varphi},\partial\boldsymbol{\varphi})$,

$$\boldsymbol{\omega}(\boldsymbol{\varphi},\partial\boldsymbol{\varphi})=\frac{\partial\boldsymbol{\varphi}+[\boldsymbol{\varphi}\partial\boldsymbol{\varphi}]}{1+\boldsymbol{\varphi}^2}. \tag{2}$$

In the exchange approximation the system Lagrangian is quadratic in $\boldsymbol{\omega}(\boldsymbol{\varphi},\partial\boldsymbol{\varphi}/\partial t)$ and $\boldsymbol{\omega}(\boldsymbol{\varphi},\partial\boldsymbol{\varphi}/\partial \mathbf{x})$,

$$L=\int dx\{A_{ij}\omega_i(\boldsymbol{\varphi},\dot{\boldsymbol{\varphi}})\omega_j(\boldsymbol{\varphi},\dot{\boldsymbol{\varphi}}) -B_{ij}\omega_i(\boldsymbol{\varphi},\boldsymbol{\varphi}')\omega_j(\boldsymbol{\varphi},\boldsymbol{\varphi}')\}, \tag{3}$$

where the dot and prime stand for the time and position derivatives, respectively. The symmetry of the tensors A_{ij} and B_{ij} is determined by the specific sublattice structure of the magnetic material (for details see Ref. 12). Note that in view of the non-Abelian nature of the SO(3) group the Lagrangian (3) proves to be essentially nonlinear, which explains the existence of soliton solutions even for the bilinear-in-ω Lagrangian (3). The same fact explains the non-Euclidean nature of the space of the field variable $\boldsymbol{\varphi}$ determining a parametrization of the SO(3) group, which leads to solitons with nontrivial topological properties (see below).

In this paper we examine a simple model. We take a one-dimensional magnetic material and assume that both tensors in the Lagrangian (3), A_{ij} and B_{ij}, are proportional to the unit tensor. In this case

$$L=\frac{\chi a^2}{2g^2}\int dx\{\boldsymbol{\omega}^2(\boldsymbol{\varphi},\dot{\boldsymbol{\varphi}})-c^2\boldsymbol{\omega}^2(\boldsymbol{\varphi},\boldsymbol{\varphi}')\}, \tag{4}$$

where χ is the magnetic susceptibility, a is the transverse size of the system (or order of the lattice constant), g is the gyromagnetic ratio, and c is the velocity of spin waves in the linear theory.

The most widespread representation of the field variable $\boldsymbol{\varphi}$ is in the form $\boldsymbol{\varphi}=\mathbf{n}\tan\theta/2$, where θ is the value of the rotation angle, and $\mathbf{n}$ is the unit vector in the direction of rotation, $\mathbf{n}^2=1$. With this parametrization the structure of the rotation matrix R_{ij} and the Cartan forms $\boldsymbol{\omega}(\boldsymbol{\varphi},\partial\boldsymbol{\varphi})$ is determined by the following relationships:

$$\begin{gathered}R_{ij}=\delta_{ij}+(1-\cos\theta)(\mathbf{n}_i\mathbf{n}_j-\delta_{ij})+\varepsilon_{ikj}\mathbf{n}_k\sin\theta,\\ \omega_i=\mathbf{n}_i\partial\theta+\sin\theta\partial\mathbf{n}_i+(1-\cos\theta)\varepsilon_{ikj}\mathbf{n}_k\partial\mathbf{n}_j,\end{gathered} \tag{5}$$

where ε_{ijk} is the totally antisymmetric third-rank tensor.

With another parametrization of the SO(3) group, namely via a unit four-dimensional vector l_μ nonlinearly related to the variable $\boldsymbol{\varphi}$,

$$l_\mu=\left(\frac{\boldsymbol{\varphi}}{(1+\boldsymbol{\varphi}^2)^{1/2}},\frac{1}{(1+\boldsymbol{\varphi}^2)^{1/2}}\right),$$

the Lagrangian is quadratic in the components of vector l_μ and has the form typical of chiral models:

$$L=\frac{2\chi c^2a^2}{g^2}\int dx\left\{\frac{1}{c^2}\left(\frac{\partial l_\mu}{\partial t}\right)^2-\left(\frac{\partial l_\mu}{\partial x}\right)^2\right\},$$

and the nonlinearity of the system is determined by the geometric condition $l_\mu^2=1$. Note that the model is Lorentz-invariant (with a characteristic velocity c). For more general models of magnetic materials the Lorentz invariance may break down.

To treat for relativistic interactions in the Lagrangian of the model we must allow for additional terms, whose various types were discussed in Ref. 12. The relativistic term suggested by Andreev and Marchenko[12] leads to magnetic anisotropy and can be written in the form

$$\Delta L_a=-\frac{2\chi a^2}{g^2}\int dx\, w_a(\boldsymbol{\varphi}^2), \tag{6}$$

where $w_a(\boldsymbol{\varphi}^2)\approx\omega_0^2\boldsymbol{\varphi}^2$ for $\boldsymbol{\varphi}^2\ll1$. Here ω_0 corresponds to the energy of activation of spin waves emerging in such a model. According to (6), the ground state of the magnetic material corresponds to a well-defined angle $\boldsymbol{\varphi}$ ($\boldsymbol{\varphi}=0$), which means that the system's symmetry with respect to homogeneous spin rotations can break and that this state conforms to models in which the equilibrium directions of the spins are related to selected directions in the crystal.

A remark is in order. Strictly speaking, such a model can be used only to describe amorphous magnetic materials of the spin glass type,[12] and the results obtained on the basis of the Lagrangian (4) can be used only in studies of 1D magnetic materials with randomly frozen spins (such materials were investigated in Ref. 12). With magnetic materials that have a finite number of sublattices (of the UO_2 and $YMnO_3$ types), the tensors A_{ij} and B_{ij} in the Lagrangian (3) contain a large number of independent components, in view of which there emerge two different characteristic velocities c, one for longitudinal waves and the other for transverse waves.[12] Furthermore, with such generalization some integrals of motion discussed below [see Eqs. (15)] become invalid.

On the other hand, for magnetic materials with pronounced anisotropy in the arrangement of the spins (helical magnetic materials and magnetic materials with triangular magnetic structure), the different directions of spin rotation are nonequivalent, and scalar models that allow for rotation only about one, easy-type, axis come into play. Such models were discussed in Refs. 8, 10, and 15. What is lost is the main feature of noncollinear antiferromagnets caused by the need to use a multicomponent dynamic variable that parametrizes the complete SO(3) group. To emphasize this specific feature, we restrict our analysis to a model that has the highest symmetry and is described by the Lagrangian (4); we also allow for the anisotropy energy in the simplest isotropic form (6). We discuss what statements become invalid when the model is generalized.

3. SOLITON SOLUTIONS: GENERAL ANALYSIS AND QUASI-CLASSICAL QUANTIZATION

Putting $\boldsymbol{\varphi}=\mathbf{n}\varphi$, we can write the Lagrangian in terms of the variables φ and $\mathbf{n}$:

$$L=\frac{2\chi a^2}{g^2}\int dx\left\{\left(\frac{\dot\varphi}{1+\varphi^2}\right)^2+\frac{\varphi^2\dot{\mathbf{n}}^2}{1+\varphi^2}-c\left[\left(\frac{\varphi'}{1+\varphi^2}\right)^2+\frac{\varphi^2\mathbf{n}'^2}{1+\varphi^2}\right]-w_a(\varphi)^2\right\}. \tag{7}$$

Clearly, the general equation for the variable $\boldsymbol{\varphi}$ has a class of solutions of the form

$$\boldsymbol{\varphi}(x,t)=\varphi(x)\mathbf{n}(t),\quad \mathbf{n}^2=1. \tag{8}$$

In a soliton of this type the unit vector precesses about an arbitrary constant vector $\boldsymbol{\omega}$, i.e., $\dot{\mathbf{n}}=[\boldsymbol{\omega}\mathbf{n}]$.

It is convenient to write the equation for $\varphi(x)$ in terms of the variable θ, with $\varphi=\tan\theta/2$:

$$c^2\theta''+\omega^2\sin\theta-\frac{dw_a}{d\theta}=0. \tag{9}$$

The first integral of Eq. (9) is

$$c^2\left(\frac{d\theta}{dx}\right)^2+2\omega^2(1-\cos\theta)-w_a(\theta)=\text{const}, \tag{10}$$

and the structure of the solution can be established in quadratures for any type of function $w_a(\theta)$. Since in view of (6) $w_a(\theta)\to\omega_0^2\theta^2$ as $\theta\to 0$, the desired localized soliton solution with an exponentially decreasing function $\theta(x)$ exists for $\omega^2<\omega_0^2$. This condition is completely natural: the soliton frequency must be below the edge of the linear magnon spectrum.

The coefficients of Eq. (10) are periodic with a minimum period of 2π and do not change when $2\pi-\theta$ is substituted for θ. This suggests two types of behavior of the function $\theta(x)$. One corresponds to the angle θ being unfolded to 2π, say

$$\theta(-\infty)=0,\quad \theta(+\infty)=2\pi,\quad \frac{d\theta}{dx}(\pm\infty)=0. \tag{11}$$

The states with $\theta=0$ and $\theta=2\pi$ correspond to the same value of the variable φ and are physically identical in view of the continuity of the solution $\theta=\theta(x)$. In such a soliton there is always a point $x=x_0$ at which the value $\theta=\pi$ is attained, which corresponds to a discontinuity in the field φ of the form

$$\boldsymbol{\varphi}=\varphi(x)\mathbf{n}_0,\quad \varphi(x_0\pm\varepsilon)\to\pm\infty\quad \text{as}\quad \varepsilon\to 0, \tag{12}$$

where $\mathbf{n}_0$ is an arbitrary unit vector (note that $\mathbf{n}_0$ can even vary with time, $\mathbf{n}_0=\mathbf{n}_0(t)$). Since rotations through an angle π about the axes $\mathbf{n}_0$ and $-\mathbf{n}_0$ are identical, this discontinuity in the field φ does not disrupt the continuity of the state of the magnetic material. Since such a soliton corresponds to a contour that cannot be contracted to a point (see Ref. 5), the soliton of the first type has a nonzero topological charge (the principles of topological analysis of solitons were developed by Volovik and Mineev[16,17]).

For solitons of the second type the values of θ from left and right coincide, $\theta(-\infty)=\theta(+\infty)=0$, and in view of the above-mentioned symmetry properties of Eq. (10) the value $\theta=\pi$ is not reached and the function $\varphi(x)$ is continuous. Such a soliton corresponds to a contour in the space of φ that can be contracted to a point, i.e., the topological charge of such a soliton is zero.

A system with the Lagrangian (7) has the usual constants of motion: the energy

$$E=\frac{2\chi a^2}{g^2}\int dx\left\{\left(\frac{\dot{\varphi}}{1+\varphi^2}\right)^2+\frac{\varphi^2\dot{\mathbf{n}}^2}{1+\varphi^2}+c^2\left[\left(\frac{\varphi'}{1+\varphi^2}\right)^2+\frac{\varphi^2\mathbf{n}'^2}{1+\varphi^2}\right]+w_a(\varphi^2)\right\}, \tag{13}$$

and the momentum

$$P=-\int dx\frac{\partial L}{\partial\dot{\boldsymbol{\varphi}}}\boldsymbol{\varphi}'=-\frac{4\chi a^2}{g^2}\int dx\left\{\frac{\dot{\varphi}\varphi'}{(1+\varphi^2)^2}+\frac{\varphi^2(\dot{\mathbf{n}}\mathbf{n}')}{1+\varphi^2}\right\}. \tag{14}$$

In the exchange approximation the system has two more vector constants of motion: the spin angular momentum $\mathbf{M}$ and an additional integral of motion $\mathbf{N}$,

$$\mathbf{M}=\frac{4\chi}{g^2}\int dx\frac{\dot{\boldsymbol{\varphi}}+[\boldsymbol{\varphi}\dot{\boldsymbol{\varphi}}]}{1+\varphi^2},\quad \mathbf{N}=\frac{4\chi}{\hbar g^2}\int dx\frac{[\boldsymbol{\varphi}\dot{\boldsymbol{\varphi}}]}{1+\varphi^2}, \tag{15}$$

related to the symmetry of the O(4) group (Planck's constant in the definition of $\mathbf{N}$ is added for convenience). If relativistic interactions are considered [Eq. (6)], the angular momentum ceases to be conserved, but $\mathbf{N}$ remains a constant of motion of the system ($\mathbf{N}$ ceases to be a constant of motion when exchange energy of a more general type with nonunit tensors A_{ij} and/or B_{ij} is taken into account).

The presence of an additional constant of motion makes the existence of dynamic two-parameter solitons possible. In the case of uniaxial ferromagnets[18] and antiferromagnets[19] an additional constant of motion that ensures the stability of solitons has the simple meaning of the projection of the total magnetization on the preferred axis M_z and can be directly used for quasi-classical quantization of solitons: the z projection of the total spin of a soliton, equal to $M_z/2\mu_0$, where μ_0 is the Bohr magneton, must have an integral value. The meaning assigned to this value may be the number of magnons bound in a soliton (see, e.g., Ref. 18). Below we show that one of the projections of $\mathbf{N}$ plays a similar role in the quasi-classical quantization of solitons in the model of a multiple-sublattice magnetic material considered here.

To this end it has proved convenient to introduce angular variables parametrizing the unit vector $\mathbf{n}$:

$$\mathbf{n}=(\cos\alpha\cos\beta,\ \cos\alpha\sin\beta,\ \sin\alpha). \tag{16}$$

In terms of the variables θ, α, and β the system Lagrangian assumes the form

$$L=\frac{\chi a^2}{2g^2}\int dx\left\{\dot{\theta}^2+4\sin^2\frac{\theta}{2}(\dot{\alpha}^2+\dot{\beta}^2\cos^2\alpha)-c^2\left[\theta'^2+4\sin^2\frac{\theta}{2}(\alpha'^2+\beta'^2\cos^2\alpha)\right]-w_a\right\}. \tag{17}$$

Selecting the constant vector $\boldsymbol{\omega}$ so that it is directed along the Z axis, we find that a soliton corresponding to Eqs. (8) and (9) has $\varphi=\varphi(x)$, $\alpha=0$, and $\beta=\omega t+\beta_s$, where β_s is the initial phase. The structure of the moving soliton can be obtained from (8) by employing the Lorentz transformation, i.e., by introducing the following substitutions: $t\to\tau=(t-xV/c^2)/\gamma$ and $x\to\xi=(x-Vt)/\gamma$, with $\gamma=(1-V^2/c^2)^{-1/2}$.

For comparison let us discuss the properties of linear spin waves in the model (1). As shown in Ref. 12, in the linear approximation the equations of motion for φ describe three degenerate spin-wave branches of the form

$$\boldsymbol{\varphi}=\boldsymbol{\varphi}_0\exp[i(kx-\omega t)] \tag{18}$$

with a dispersion law $\omega=(\omega_0^2+c^2k^2)^{1/2}$, where $\boldsymbol{\varphi}_0$ is a constant vector, and $\varphi_0^2\to 0$. In view of the degeneracy of the spectrum, both linearly and circularly polarized waves can be taken as the natural waves in the system. For instance, we can take a wave linearly polarized along the Z axis, $\boldsymbol{\varphi}\propto\mathbf{e}_z\cos(kx-\omega t)$, and two circularly polarized waves in which $\boldsymbol{\varphi}_0\perp\mathbf{e}_z$,

$$\varphi_x=\varphi_0\cos(\omega t-kx),\quad \varphi_y=\varphi_0\sin(\omega t-kx), \tag{19}$$

which corresponds to $\alpha=0$ and $\beta=kx-\omega t$ in our notation. These circularly polarized waves are the linear limit of the soliton (8).

Let us establish the meaning of the constant of motion $\mathbf{N}$ for linear spin waves and for solitons. To this end we find the momenta that are the canonical conjugates of the angular variables α and β:

$$P_\alpha=\frac{4\chi}{g^2}\frac{\varphi^2}{1+\varphi^2}\dot{\alpha},\quad P_\beta=\frac{4\chi}{g^2}\frac{\varphi^2}{1+\varphi^2}\dot{\beta}\sin^2\alpha. \tag{20}$$

Since the Lagrangian (17) is cyclic in the variable β, the integral

$$J_\beta=\frac{1}{2\pi}\frac{4\chi a^2}{g^2}\int dx\int_0^{2\pi}d\beta\frac{\varphi^2\sin^2\alpha}{1+\varphi^2}\dot{\beta}, \tag{21}$$

which has the meaning of action, is conserved. Clearly, $\dot{\beta}\sin^2\theta=[\mathbf{n}\dot{\mathbf{n}}]$, i.e., according to the definition (15) for $\mathbf{N}$ we have $J_\beta=\hbar N_z$. Thus, the quantity N_z can be interpreted as a quantum number. It is easy to show that for a linear spin wave we have $E=\hbar\omega(k)$ and $P=\hbar k N_z$, with the result that

$$N=\frac{4\chi a^2}{\hbar g^2}\int dx\frac{\varphi^2}{1+\varphi^2}[\mathbf{n}\dot{\mathbf{n}}] \tag{22}$$

coincides with the number of the magnons of the spin wave with the circular polarization (19) (since for a delocalized wave the integral (22) is proportional to the size of the system, it is better to speak of the magnon density). For a soliton solution, $\varphi(x)$ rapidly decreases and the number N of magnons in the soliton is finite. Here for a localized soliton with a given N the energy E is lower than $\hbar\omega_0 N$, so that such a soliton can be interpreted as a bound state of many magnons in a soliton. Note the marked difference between this model and a ferromagnet, in which the finite value of the integral N may correspond only to a nontopological localized soliton, while an infinite value of N corresponds to a topological soliton (what is known as a π-kink).

The Lorentz invariance of the system considered makes it possible, by starting with solutions of type (8), to easily construct a solution corresponding to a moving soliton. It is found that the additional integral of motion $N=N(\omega)$ expressed in terms of the frequency of precession in the reference frame at rest remains invariant under Lorentz transformations, i.e., $N(\omega,V)=N(\omega,0)=N(\omega)$. Hence instead of the parameter ω (the frequency in the reference frame in which the soliton is at rest) we can use $N(\omega)$, the number of magnons bound in the soliton, to classify solitons.

Plugging (10) into (13) and (14) and using the equations of motion, we arrive at the following expressions for the two other constants of motion:

$$E=\frac{E_0}{(1-V^2/c^2)^{1/2}},\quad \mathbf{P}=\frac{\mathbf{V}}{c^2}\frac{E_0}{(1-V^2/c^2)^{1/2}}, \tag{23}$$

where E_0 is the energy of a precessing soliton at rest. Note that the relationship between $\mathbf{P}$, $\mathbf{V}$, and E for a soliton is the same as for a particle in relativistic mechanics. Furthermore, it can be shown that if the soliton energy is considered a function of the soliton momentum and the integral N, the following relations hold:

$$V=\frac{\partial E(P,N)}{\partial P},\quad \hbar\omega=\frac{\partial E(P,N)}{\partial N}. \tag{24}$$

Note that in deriving Eqs. (23) and (24) we did not use the specific form of the anisotropy energy w_a. The only requirement here is that the form of w_a allow for the existence of soliton solutions with a rapidly decreasing function $\varphi(x)$.

4. THE SPECIFIC SOLITON STRUCTURE

To determine the soliton structure we must specify the anisotropy energy w_a. For the simple model considered here (a completely isotropic distribution of spins, as in a spin glass), we follow Ref. 12 and write this energy in the form

$$w_a(\theta)=\omega_0^2\left[b(1-\cos\theta)+\frac{1}{2}(1-b)\sin^2\theta\right], \tag{25}$$

where ω_0 corresponds to the activation energy of the linear spin waves. Note that the ground state with $\varphi=0$ is stable under small perturbations for $\omega_0^2>0$, irrespective of the sign and value of the parameter b. Below we restrict our analysis to the case with $b>0$.

Proceeding from the Lagrangian (17) and the anisotropy energy (25), we arrive at the equations of motion for the variables θ, α, and β in the following form:

$$\begin{aligned}
&c^2\theta''+\ddot{\theta}+\sin\theta(\dot{\alpha}^2+\dot{\beta}^2\cos^2\alpha)-c^2\sin^2\theta(\alpha'^2\\
&\qquad+\beta'^2\cos^2\alpha)-\omega_0^2\left[b\sin\theta+\frac{1-b}{2}\sin 2\theta\right]=0,\\
&c^2\left(\alpha'\sin^2\frac{\theta}{2}\right)'-\left(\dot{\alpha}\sin^2\frac{\theta}{2}\right)^{\cdot}+\sin\alpha\cos\alpha\sin^2\frac{\theta}{2}\\
&\qquad\times(c^2\beta'^2-\dot{\beta}^2)=0,\\
&c^2\left(\beta'\sin^2\frac{\theta}{2}\cos^2\alpha\right)'-\left(\dot{\beta}\sin^2\frac{\theta}{2}\cos^2\alpha\right)^{\cdot}=0.
\end{aligned} \tag{26}$$

A two-parameter soliton, whose general structure has been studied above, corresponds to $\alpha=0$, $\beta=\omega t-kx+\beta_s$, and $\theta=\theta(\xi)$, where $\xi=x-Vt$, $k=V\omega/c^2$, and the function $\theta(\xi)$ satisfies the equation

$$\frac{x_0^2}{\gamma^2}\frac{d^2\theta}{d\xi^2}+(\Omega-b)\sin\theta+(b-1)\sin\theta\cos\theta=0, \tag{27}$$

where we have introduced the notation $x_0=c/\omega_0$ and $\Omega=(\omega/\gamma\omega_0)^2$.

A soliton solution exists only for $\omega^2<(\gamma\omega_0)^2$, i.e., $\Omega<1$. The explicit form of the solution depends on the parameter b and the precession frequency ω. For $\Omega<b$ $(\omega^2<b(\gamma\omega_0)^2)$, the values of $\theta(-\infty)$ and $\theta(+\omega)$ are sure to differ by 2π and the soliton is a topological one (i.e., has a nonzero topological charge). The distribution corresponding to such a soliton is

$$\tan\frac{\theta}{2}=\left(\frac{1-\Omega}{b-\Omega}\right)^{1/2}\operatorname{cosech}\frac{\gamma(1-\Omega)^{1/2}(x-Vt-x_s)}{x_0}, \tag{28}$$

where x_s is an integration constant whose meaning is the coordinate of the center of the soliton at time t. For $b>1$

such topological solitons, which can be called kinks, occur at all admissible frequencies. But for $b<1$, kinks occur only at fairly low frequencies, $\omega^2<b(\gamma\omega_0)^2<(\gamma\omega_0)^2$. If the frequency exceeds $b(\gamma\omega_0)^2$ $(b<\Omega<1)$, then the soliton has no topological charge, with

$$\tan\frac{\theta}{2}=\left(\frac{1-\Omega}{\Omega-b}\right)^{1/2}\operatorname{sech}\frac{\gamma(1-\Omega)^{1/2}(x-Vt-x_s)}{x_0}, \quad (29)$$

$\theta(-\infty)=\theta(+\infty)$, and the value $\theta=\pi$ is not reached.

For $b<1$, the frequency $\omega=\omega_0\gamma b^{1/2}$ is a singular point of Eq. (27). At such a frequency the nontrivial solution of this equation determines a delocalized "domain wall" separating states with $\theta=0$ from the excited state with $\theta=\pi$. Now this wall cannot be considered a localized soliton, since it corresponds to an infinitely high soliton energy and an infinitely large number N of magnons. If we have $b<1$ and $\omega^2=b(\gamma\omega_0)^2(1\pm\delta)$, $\delta\to0$, the soliton consists of two such walls separated by a large distance of order $x_0\ln(1/|\delta|)$. In the limit $\omega\to\omega_0$, the amplitude of the dynamic soliton tends to zero, $\theta(0)\propto(\omega_0^2-\omega^2)^{1/2}$, and the soliton degenerates.

For $b>1$, a topological soliton solution exists for all frequency values that obey the inequality $\omega^2<(\gamma\omega_0)^2$. In the limit $\omega\to\gamma\omega_0$, the solution (28) becomes an algebraic soliton:

$$\tan\frac{\theta}{2}=\frac{x_0}{\gamma(x-Vt-x_s)} \quad \text{for} \quad \omega\to\gamma\omega_0, \quad b>1. \quad (30)$$

The value of the constant of motion N for solitons specified by (28) and (29) is given by the following expression (the reader will recall that at a given frequency ω this value is independent of the soliton velocity V):

$$N(\omega)=\frac{8\chi\omega x_0a^2}{g^2|b-1|^{1/2}}\begin{cases}\arctan\left(\frac{b-1}{1-\Omega}\right)^{1/2} & \text{if} \quad b>1,\\ \operatorname{Arcoth}\left(\frac{1-b}{1-\Omega}\right)^{1/2} & \text{if} \quad b<1.\end{cases} \quad (31)$$

We see that for $b<1$ the function $N(\omega)$ is nonmonotonic: for $\omega\to0$ and $\omega\to\gamma\omega_0$ we have $N(\omega)\to0$, while for $\omega^2\to b(\gamma\omega_0)^2$ the value of $N(\omega)$ increases indefinitely. But if $b<1$ holds, the function $N(\omega)$ monotonically increases with ω, reaching its maximum value at $\omega=\gamma\omega_0$ (at the point where the soliton becomes algebraic), with $dN/d\omega\to\infty$.

The value of the soliton energy in all cases can be written as

$$E(V,\omega)=\gamma\left\{\frac{4\chi ca^2}{g^2}(\omega_0^2-\omega^2)^{1/2}+\frac{1}{2}\hbar N(\omega)\left(\frac{b\omega_0^2}{\omega}+\omega\right)\right\}. \quad (32)$$

If $b=1$ holds, the topological soliton simplifies considerably:

$$\tan\frac{\theta}{4}=\exp\frac{\gamma(1-\Omega)^{1/2}(x-Vt-x_s)}{x_0},$$

and the expressions for the constants of motion N and E assume the form

$$N(\omega)=\frac{8\chi c\omega}{\hbar g^2(\omega_0^2-\omega^2)^{1/2}},$$
$$E(V,\omega)=\frac{8\chi c\omega_0^2}{g^2(\omega_0^2-\omega^2)^{1/2}(1-V^2/c^2)^{1/2}}. \quad (33)$$

In this case, for a soliton at rest ($V=0$) we can write a simple formula for the N-dependence of E:

$$E=E_0\left[1+\left(\frac{N}{N_0}\right)^2\right]^{1/2}, \quad E_0=\hbar\omega_0N_0, \quad N_0=\frac{8\chi ca^2}{\hbar g^2}. \quad (34)$$

Clearly, Eq. (34) shows that the lowest soliton energy E_0 corresponds to $N=0$, and the states with $N\neq0$ can be interpreted as excited soliton states. The characteristic value of E_0 can be estimated by setting $\xi\approx(2\mu_0)^2/a^3J$ and $c\sim Ja/\hbar$ (J is the exchange integral), assuming that the atomic spin S is unity. With these data we obtain $N_0\sim1$ and $E_0\sim\hbar\omega_0$ [the latter relationship, which is important for analyzing the soliton contribution to the thermodynamic characteristics of the system, is determined more accurately below, in Eq. (47)]. The scaling that $N_0\sim1$ means that quantum effects may play an important role in the dynamics of internal modes in a soliton.[20] Note that the corresponding value for ferromagnets is much larger than unity, with the result that the dynamics of internal modes in a soliton is quasi-classical.[18] A detailed study of the quantum properties of solitons can be done along the same line of reasoning as for collinear antiferromagnets.[20] Such analysis, however, lies outside the scope of the present investigation.

5. CALCULATING THE SOLITON DENSITY

Because the energy of solitons in quasi-one-dimensional systems is finite, at finite temperature T the thermodynamic-equilibrium density n_s of these nonlinear excitations is nonzero. Since $E_0\sim\hbar\omega_0$, the soliton density is comparable to the magnon density in the linear theory (this definitely sets antiferromagnets apart from ferromagnets, where we have $E_0\gg\hbar\omega_0$ and n_s is small).

For the case of two-parameter solitons considered in this paper, the thermodynamic-equilibrium soliton density n_s in the approximation of an ideal soliton gas with no soliton–magnon interaction is

$$n_s=\frac{1}{L}\int\int dp\, dJ_\beta w(p,J_\beta),$$
$$w(p,J_\beta)=\frac{2\pi L}{(2\pi\hbar)^2}\exp\left(-\frac{E}{T}\right), \quad (35)$$

where L is the length of the system, and $E=E(p,J_\beta)$ is the soliton energy written as a function of the generalized momenta p and J_β, the canonical conjugates of the two generalized coordinates, x_s and β_s, which determine the soliton's coordinate and phase.

In calculating the integrals in (35) it is convenient to go from integration with respect to the momentum $J_\beta=\hbar N$ to integration with respect to E. This combined with the second relationship in (24) yields the following expression for n_s in the low-temperature limit ($T\ll E_0$) of interest to us:

$$n_s=\frac{\rho^{1/2}}{x_0}\left(\frac{TE_0}{\varepsilon_0^2}\right)\exp\left(-\frac{E_0}{T}\right), \tag{36}$$

where $\varepsilon_0=\hbar\omega_0$ is the magnon activation energy, and $\rho=\rho(b)$ is a somewhat cumbersome function of parameter b (later we will see that Eq. (36) is valid if a stronger inequality holds: $T<\varepsilon_0<E_0$).

As noted earlier, Eq. (35) is written without allowing for the interaction of the soliton and the linear excitations (magnons) in the system. As demonstrated in the pioneering paper of Krumhansl and Schrieffer,[21] such an interaction leads to a characteristic interference between solitons and magnons and substantial corrections to the kink density. Currie *et al.*[22] suggested a phenomenological approach allowing for such interference. In this picture, both kinks and magnons are considered components of an ideal gas, and the soliton–magnon interaction is reduced to renormalizing the kink energy. The reason for renormalization is that when the system has a kink, (a) localized magnon modes arise, and (b) the number of magnon degrees of freedom decreases and the density of magnon states diminishes. In the approximation in which the kink density is low, the variation of the system's free energy is additive, and for this reason Currie *et al.*[22] suggested considering this variation a contribution to the kink energy and "forgetting" about the soliton–magnon interaction entirely. Here the thermodynamic-equilibrium kink density n_s is still determined by (35) but E is replaced by $E+\Sigma$, where Σ is the effective kink free energy variation caused by the interaction with magnons. Such an approach became known as "soliton phenomenology."

Thus, to determine the effect of soliton–magnon interaction on the kink density we must analyze the spectrum of linear excitations in the system with a kink and calculate the contribution to the free energy related, first, to the variation in the density of magnon states and, second, to the emergence of additional modes localized at the soliton. Hence to calculate Σ we must analyze the magnon spectrum superposed on the soliton background. It can easily be shown that at low temperatures the main contribution to the integral in (35) is provided by solitons with low velocities and precession frequencies. Consequently, it is sufficient to calculate the magnon spectrum superposed on the kink at rest, for which $\omega=V=0$.

Setting $\theta=\theta_s(x)+\vartheta(x,t)$ in the equations of motion (26), where $\theta_s(x)$ corresponds to a soliton, and assuming that ϑ, α, and β are much smaller than unity, we linearize these equations in the functions $\vartheta(x,t)$, $\alpha(x,t)$, and $\beta(x,t)$. The result is a system of three separate equations:

$$c^2\vartheta''-\ddot{\vartheta}-\omega_0^2[\cos\theta_s+(1-b)\cos 2\theta_s]\vartheta=0,$$

$$c^2\left(\alpha'\sin^2\frac{\theta_s}{2}\right)'-\left(\dot{\alpha}\sin^2\frac{\theta_s}{2}\right)^{\cdot}=0, \tag{37}$$

$$c^2\left(\beta'\sin^2\frac{\theta_s}{2}\right)'-\left(\dot{\beta}\sin^2\frac{\theta_s}{2}\right)^{\cdot}=0.$$

Assuming that $\vartheta,\alpha,\beta\sim\exp(i\omega t)$ and introducing the functions $\mu=\alpha\sin(\theta_s/2)$ and $\eta=\beta\sin(\theta_s/2)$ instead of α and β, respectively, we arrive at Schrödinger equations for ϑ, μ, and η:

$$-x_0^2\vartheta''+U_1(\theta_s)\vartheta=\frac{\omega^2}{\omega_0^2}\vartheta,$$

$$-x_0^2\mu''+U_2(\theta_s)\mu=\frac{\omega^2}{\omega_0^2}\mu. \tag{38}$$

The equation for η coincides with that for μ.

The potentials $U_1(\theta_s)$ and $U_2(\theta_s)$ are determined by the soliton's structure and have the form

$$U_1=b\cos\theta_s+(1-b)\cos 2\theta_s=1-\frac{2(4-3b)}{1+b\sinh^2(x/x_0)}+\frac{8(1-b)}{[1+b\sinh^2(x/x_0)]^2},$$

$$U_2=b\cos\theta_s+(1-b)\left(\cos^2\frac{\theta_s}{2}-\frac{3}{4}\sin^2\theta_s\right)=1-\frac{2(2-b)}{1+b\sinh^2(x/x_0)}+\frac{3(1-b)}{[1+b\sinh^2(x/x_0)]^2}. \tag{39}$$

By an appropriate change of variable the differential equations (38) with the potentials (39) can be reduced to generalized Lamé equations with four regular singular points, and their solutions can be written in terms of **P**-symbols, which makes them quite complicated.[23] Hence in what follows we restrict our discussion to the particular case $b=1$. Then the potentials U_1 and U_2 simplify considerably and are reduced to a simple reflectionless potential:

$$U_1=U_2=1-\frac{2}{\cosh^2(x/x_0)}. \tag{40}$$

The spectrum and wave functions of the Schrödinger equation with the potential (40) are well-known: the potential has a single discrete level corresponding to the eigenvalue $\lambda_0=0$ and a localized eigenfunction $f_0(x)\sim\operatorname{sech}(x/x_0)$, and a continuous spectrum with $\lambda_k=1+(kx_0)^2$ and the wave functions

$$f_k(x)\sim\left(\tanh\frac{x}{x_0}-ikx_0\right)\exp(ikx). \tag{41}$$

Thus, in the model of the magnetic material considered here with $b=1$, there are three degenerate Goldstone modes with zero frequency and three degenerate delocalized branches of the spectrum with the wave functions (41) and a dispersion law $\omega^2(k)=\omega_0^2+c^2k^2$, which coincides with the dispersion law of spin waves in a magnetic material without a soliton. However, in the presence of a kink the wave function of magnons with the wave vector k becomes distorted due to the interaction with the kink, and an asymptotic phase shift $\Delta(k)$ occurs, which for the wave functions (41) is equal to $-2\arctan(kx_0)$. This in turn leads to a change in the density of magnon states:

$$\delta\rho(k)=\frac{1}{2\pi}\frac{d\Delta(k)}{dk}=\frac{x_0}{\pi(1+k^2x_0^2)}. \tag{42}$$

The variation in the magnon free energy and, hence, in the kink energy is in general[22]

$$\Sigma = T\sum_n dk\delta\rho(k)\ln\left[1-\exp\left(-\frac{\hbar\omega_n(k)}{T}\right)\right]+T\ln Z'_{\rm loc}. \tag{43}$$

The first term on the right-hand side of Eq. (43) is caused by the variation in the density of states of the continuous spectrum (n labels the various branches of the continuous spectrum), and the second by the emergence of modes localized at the kink: $Z'_{\rm loc}$ is the partition function of such modes, and the prime indicates that the levels are measured from the ground state, with the result that zero-frequency modes contribute nothing to Σ.

Since in our model of a magnetic material with $b=1$ all three localized modes are of the Goldstone type and have zero frequency and all three branches of the continuous spectrum are degenerate, we can write

$$\Sigma = 3T\int dk\delta\rho(k)\ln\left[1-\exp\left(-\frac{\hbar\omega(k)}{T}\right)\right], \tag{44}$$

where $\omega(k)=(\omega_0^2+c^2k^2)^{1/2}$.

Calculating the integral in (44) with allowance for (42) at temperatures that are low ($T\ll\varepsilon_0=\hbar\omega_0$) and high ($T\gg\varepsilon_0$) compared to the magnon energy, we find

$$\Sigma\approx 3\ln\frac{T}{\varepsilon_0}-3T\ln 2 \quad \text{for} \quad T\gg\varepsilon_0 \tag{45}$$

and $\Sigma\approx 0$ for $T\ll\varepsilon_0$. Clearly, the result for $T\gg\varepsilon_0$ is three times the contribution in the sine-Gordon model,[22] where the potential generated by the kink for the magnons is also of the form (40).

Hence at low temperatures ($T\ll\varepsilon_0$) the thermodynamic-equilibrium kink density is determined by a formula of the same type as that without soliton–magnon interference [see Eq. (36)] and $n_s\sim T\exp(-E_0/T)$, while at temperatures that are high compared to the magnon energy ($T\gg\varepsilon_0$) the power to which the temperature is raised in the pre-exponential factor changes considerably:

$$n_s=\frac{8\rho^{1/2}E_0}{\hbar c}\left(\frac{\varepsilon_0}{T}\right)^2\exp\left(-\frac{E_0}{T}\right)\sim T^{-2}\exp\left(-\frac{E_0}{T}\right). \tag{46}$$

Note that here we consider only temperatures for which the kink density is low, i.e., $T\ll E_0$. But Eq. (46) is valid for $T\gg\varepsilon_0$, and hence it is valid in the interval $\varepsilon_0<T<E_0$. For such an interval to exist the magnon activation energy ε_0 must be considerably lower than the soliton energy E_0. Comparing these two energies, we obtain

$$\frac{\varepsilon_0}{E_0}=\frac{\hbar\omega_0}{8\chi c\omega_0 a^2/g^2}=\frac{\hbar g^2}{8\chi c a^2}. \tag{47}$$

In contrast to a ferromagnet, this quantity is independent of relativistic constants and is determined solely by the exchange interaction. Using an expression for the magnon velocity c typical of antiferromagnets,[2] we find that $(\varepsilon_0/E_0)\sim 1/S$, where S is the value of the atomic spin, and the expression contains no other small constants. Hence the temperature interval in which Eq. (46) is valid exists for $S\gg 1$, i.e., in the quasi-classical limit. The same situation is true for collinear antiferromagnets.[4,20]

6. THE DYNAMIC STRUCTURE FACTOR OF KINKS

One of the most important characteristics of a magnetic system is the dynamic structure factor $G_{ij}(q,\nu)$, which determines the neutron inelastic scattering cross section. This factor is the Fourier transform in positions and time of the correlation function of the spin density $\mathbf{S}(x,t)$:

$$G_{ij}(q,\nu)=\int\int dx\,dt\exp[i(qx-\nu t)]\langle S_i(x,t)S_j(0,0)\rangle. \tag{48}$$

Here angle brackets stand for statistical averaging over the equilibrium state of the system. Naturally, in the high-frequency range the main contribution to the dynamic structure factor is determined by linear excitations (magnons). However, in the low-frequency range the main contribution to the dynamic structure factor is provided by thermodynamic-equilibrium topological solitons. In particular, as shown in Ref. 24, the latter form the central peak in the neutron scattering cross section. For this reason we are interested only in the part of the dynamic structure factor related to solitons.

If there is only one soliton in the system, statistical averaging in (48) presupposes averaging over the soliton's phase space (i.e., over the soliton's initial coordinate x_s and initial phase β_s and over the respective momenta p and J_β) with the Gibbs distribution function,

$$\langle f(x_s,p,\beta_s,J_\beta)\rangle=\frac{1}{(2\pi\hbar)^2 Z}\int_{-\infty}^{\infty}dp\int_{-\infty}^{\infty}dx_s\int_{-\infty}^{\infty}dJ_\beta \times\int_0^{2\pi}d\beta_s f(x_s,p,\beta_s,J_\beta) \times\exp\left(-\frac{E+\Sigma}{T}\right), \tag{49}$$

where T is the temperature, $E+\Sigma$ is the soliton energy renormalized by the interaction with magnons, and Z is the partition function:

$$Z=\frac{1}{(2\pi\hbar^2)}\int_{-\infty}^{\infty}dp\int_{-\infty}^{\infty}dx_s\int_{-\infty}^{\infty}dJ_\beta\int_0^{2\pi}d\beta_s f \times\exp\left(-\frac{E+\Sigma}{T}\right). \tag{50}$$

Calculating the partition function is in many respects similar to calculating the equilibrium soliton density, which we did in Sec. 5. In the low-temperature limit we are interested in ($T\ll E_0$) we arrive at the following expression:

$$Z=\begin{cases}\dfrac{L}{x_0}\rho^{1/2}\left(\dfrac{TE_0}{\varepsilon_0^2}\right)\exp\left(-\dfrac{E_0}{T}\right) & \text{if } T<\varepsilon_0<E_0,\\[2ex] \dfrac{8L}{x_0}\rho^{1/2}\left(\dfrac{E_0\varepsilon_0}{T^2}\right)\exp\left(-\dfrac{E_0}{T}\right) & \text{if } \varepsilon_0<T<E_0.\end{cases} \tag{51}$$

The spin density $\mathbf{S}(x,t)$ is a linear combination of sublattice spins $\mathbf{S}_n(x,t)$, with the result that the system acquires a large set of dynamic structure factors related to the correlators of the various components of the magnetization vectors of the different sublattices. However, in accordance with the idea of the method of effective Lagrangians as applied to magnetically ordered media, in a phenomenological description of long-wave excitations we can introduce no more than three mutually orthogonal unit vectors $\mathbf{l}_\sigma$, $\sigma=1,2,3$, instead of analyzing the complex sublattice structure, and consider rotation operations of the form (1) on these vectors.[12] Here, in the neutron scattering problem, we must deal with the correlators of the components of the three vectors $\mathbf{l}_\sigma$ and the correlators related to the magnetization vector $\mathbf{m}$. The correlators of the vectors $\mathbf{m}$ determine the central peak for a zero wave vector of the neutrons, and the correlators of the vectors $\mathbf{l}_\sigma$, which act as distinctive antiferromagnetism vectors, cause the appearance of the central peak near the reciprocal lattice vector $Q_B=\pi/a$, with a the lattice constant (see, e.g., Ref. 3 and 25).

We start with the correlator of the $\langle m_i(x,t)m_j(0,0)\rangle$ type. For a two-sublattice antiferromagnet the corresponding dynamic structure factor was calculated in Ref. 26. In accordance with (15), the magnetic moment density is

$$\mathbf{m}=\frac{\chi}{g}\{\mathbf{n}\dot{\theta}+\dot{\mathbf{n}}\sin\theta+(1-\cos\theta)[\mathbf{n}\dot{\mathbf{n}}]\}, \tag{52}$$

and the class of soliton solutions (8) with $\alpha=0$ $(n_z=0)$ considered above corresponds to

$$m_x=\frac{\chi}{g}(\dot{\theta}\cos\beta-\omega\sin\beta\sin\theta),$$
$$m_y=\frac{\chi}{g}(\dot{\theta}\sin\beta+\omega\cos\beta\sin\theta), \tag{53}$$
$$m_z=\frac{\chi}{g}\omega(1-\cos\theta).$$

Clearly, the nonzero dynamic structure factors are $G_{zz}^{(m)}$ and $G_{\pm}^{(m)}$ where $G_{\pm}^{(m)}=\langle m_+m_-\rangle_{q,\nu}$, with $m_\pm=m_x\pm im_y$.

Plugging the distribution of magnetization in a soliton [Eq. (28)] into (53) and the correlator (49) and calculating the Fourier transforms, we arrive at the following expression for the dynamic structure factor $G_{zz}^{(m)}(q,\nu)$ at low temperatures $(T\ll E_0)$:

$$G_{zz}^{(m)}(q,\nu)=\frac{1}{L}\frac{2^{7/2}\pi^{5/2}\chi^2}{\rho g^2}\frac{|q|c^3}{\omega_0^2\tilde{\gamma}^{1/2}\sinh^2(\pi q x_0/2\tilde{\gamma})}\times\frac{T}{E_0}\exp\left[-\frac{E_0}{T}(\tilde{\gamma}-1)\right], \tag{54}$$

where $\tilde{\gamma}=(1-\nu^2/q^2c^2)^{-1/2}$. In many respects the structure of $G_{\pm}^{(m)}(q,\nu)$ is similar to that of (54) but is much more involved, so that we do not give it here.

If the system contains N_s solitons instead of one but the soliton density $n_s=N_s/L$ is low, the corresponding structure factor can be obtained from (54) by replacing the factor $1/L$ with n_s.

What is important is that all the dynamic structure factors determined by correlators of the $\langle m_i m_j\rangle$ type are small since they are proportional to the susceptibility χ of the magnetic material, $\chi\ll 1$, with the result that the amplitude of the central peak near the zero wave vector, the amplitude proportional to these dynamic structure factors, is fairly low.

The dynamic structure factors related to the correlators of the vectors $\mathbf{l}_\sigma$ are expressed, in accordance with (1), in terms of the correlators of the different components $R_{ij}(x,t)$ of the rotation matrix as follows:

$$G_{ijkl}^{(l)}(q+Q_b,\nu)=\int\int dx\,dt\,\exp\{i(qx-\nu t)\}\times\langle R_{ij}(x,t)R_{kl}(0,0)\rangle. \tag{55}$$

These dynamic structure factors contain no such small factor as χ and hence the amplitude of the magnetic Bragg peak is much higher than the amplitude of the peak near $q=0$. The general structure of $G_{ijkl}^{(l)}$ is similar to that of (54), but the temperature dependence is somewhat different:

$$G_{ijkl}^{(l)}\sim T^{1/2}\exp\left[-\frac{E_0}{T}(\tilde{\gamma}-1)\right]. \tag{56}$$

Summing up, we can say that the properties of the soliton gas in multiple-sublattice noncollinear antiferromagnets differ considerably from the corresponding properties of magnetic materials with a collinear structure, both ferromagnets and antiferromagnets. These differences manifest themselves in a number of observable characteristics of quasi-one-dimensional magnetic materials, and not only in the dynamic structure factors but also in such a statistical characteristic observed in neutron scattering experiments as the correlation radius ξ of the soliton gas, $\xi=1/2n_s$. Clearly, for $T<E_0$ the main (exponential) temperature dependence of ξ is universal for both collinear and noncollinear magnetic materials, $\xi\sim T^p\exp(E_0/T)$, but the temperature dependences of the pre-exponential factor for these types of magnetic materials are different: $p=2$ for the noncollinear magnetic materials considered here [see Eq. (46)], but $p=1$ for collinear antiferromagnets (see Refs. 4, 20, and 25).

This research was done with support from the International Science Foundation (Grant UB7200). One of the authors (B.A.I.) is grateful for the help from the International Soros Program of Supporting Education in Exact Sciences (the Revival Fund), Grant No. 25.

[1] Yu. A. Izyumov, Usp. Fiz. Nauk **155**, 553 (1988) [Sov. Phys. Usp. **31**, 689 (1988)].

[2] V. G. Bar'yakhtar and B. A Ivanov, in *Soviet Scientific Reviews, Section A*, Vol. 16, No. 4 (1993).

[3] H.-J. Mikeska and M. Steiner, Adv. Phys. **40**, 191 (1991).

[4] B. A. Ivanov and A. K. Kolezhuk, Fiz. Nizk. Temp. **21**, 355 (1995) [Low Temp. Phys. **21**, 275 (1995)].

[5] B. A. Ivanov, Solid State Commun. **34**, 437 (1980).

[6] Yu. A. Beletskiĭ, B. A. Ivanov, and A. L. Sukstanskiĭ, Teoret. Mat. Fiz. **83**, 163 (1990).

[7] I. A. Akhiezer and D. P. Belozerov, Fiz. Nizk. Temp. **10**, 544 (1984) [Sov. J. Low Temp. Phys. **10**, 285 (1984)].

[8] E. P. Stefanovskiĭ and A. L. Sukstanskiĭ, Zh. Éksp. Teor. Fiz. **104**, 3434 (1993) [JETP **77**, 628 (1993)].

[9] A. L. Sukstanskiĭ, Z. Phys. B **86**, 69 (1992).

[10] M. Wintel, H. U. Evert, and W. Apel, Europhys. Lett. **25**, 711 (1994).

[11] E. N. Khats'ko, A. S. Chernyĭ, and A. I. Kaplienko, Fiz. Nizk. Temp. **20**, 1029 (1994) [Low Temp. Phys. **20**, 811 (1994)].
[12] A. F. Andreev and V. I. Marchenko, Usp. Fiz. Nauk **130**, 39 (1980) [Sov. Phys. Usp. **23**, 21 (1980)].
[13] D. V. Volkov, A. A. Zheltukhin, and Yu. P. Bliokh, Fiz. Tverd. Tela (Leningrad) **13**, 1668 (1971) [Sov. Phys. Solid State **13**, 1396 (1971)].
[14] D. V. Volkov and A. A. Zheltukhin, Zh. Éksp. Teor. Fiz. **78**, 1867 (1980) [Sov. Phys. JETP **51**, 937 (1980)].
[15] A. L. Sukstanskiĭ and B. A. Ivanov, Zh. Éksp. Teor. Fiz. **108**, 914 (1995) [JETP **81**, 502 (1995)].
[16] G. E. Volovik and V. P. Mineev, Zh. Éksp. Teor. Fiz. **72**, 2256 (1977) [Sov. Phys. JETP **45**, 1186 (1977)].
[17] G. E. Volovik and V. P. Mineev, Zh. Éksp. Teor. Fiz. **73**, 767 (1977) [Sov. Phys. JETP **46**, 401 (1977)].
[18] A. M. Kosevich, B. A. Ivanov, and A. S. Kovalev, *Nonlinear Magnetization Waves. Dynamic and Topological Solitons*, Naukova Dumka, Kiev (1983) [in Russian].
[19] I. V. Bar'yakhtar and B. A. Ivanov, Fiz. Nizk. Temp. **3**, 544 (1977).
[20] B. A. Ivanov and A. K. Kolezhuk, Phys. Rev. Lett. **76**, 511 1996.
[21] J. A. Krumhansl and J. R. Schrieffer, Phys. Rev. B **11**, 3535 (1975).
[22] J. R. Currie, J. A. Krumhansl, A. R. Bishop, and S. E. Trullinger, Phys. Rev. B **22**, 477 (1980).
[23] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge Univ. Press, Cambridge (1944).
[24] H.-J. Mikeska, J. Phys. C **11**, L29 (1978).
[25] Yu. A. Izyumov and N. A. Chernoplekov, in *Neutrons and Solids*, Vol. 3, Energoizdat, Moscow (1983) [in Russian].
[26] K. Sasaki, Jap. J. Phys. **53**, 2872 (1984).

Translated by Eugene Yankovsky