Emission of low-frequency radiation by electrons moving along a standing laser wave

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The radiation of harmonics during the interaction of nonrelativistic electrons with a powerful standing laser wave is studied. The number of photons per pump pulse is calculated in two limiting cases: electron energy much greater than and close to the amplitude of the ponderomotive potential. It is shown that this effect can be observed experimentally with modern laser technology. [S1063-7761(96)00612-9]

1. INTRODUCTION

The interaction of an electron with a strong traveling electromagnetic wave was first described in Ref. 1. Multiphoton Compton scattering in the field of a traveling plane wave was studied in Ref. 2. The problem of the elastic scattering of nonrelativistic electrons in the field of a standing electromagnetic wave was first studied by Kapitza and Dirac.[3] The experimental observation of this effect was described in Ref. 4. The Kapitza–Dirac effect in the field of a strong standing electromagnetic wave was investigated in Refs. 5 and 6. The modulation and acceleration of a beam of nonrelativistic electrons in the field of two counterpropagating waves were studied in Refs. 7 and 8. In Ref. 7 it was shown, specifically, that the motion of an electron, averaged over fast oscillations, in the field of a standing wave can be described with the aid of a ponderomotive potential (Gaponov–Miller potential).

The motion of an electron in this potential can be accompanied not only by elastic scattering but also by spontaneous emission, whose intensity was first estimated in Refs. 9 and 10. In Ref. 9 an expression for the emission probability was obtained in first-order perturbation theory, when the parameter $m=V/\varepsilon_{\parallel}$ (V_0 is the amplitude value of the ponderomotive potential, $\varepsilon_{\parallel}=p_{\parallel}^2/2m_e$ is the electron energy associated with the motion of an electron along the standing wave) was assumed to be small: $m\ll 1$. A different limiting case was studied in Ref. 10: The electron kinetic energy $\varepsilon_{\parallel}$ was only slightly greater than the amplitudes V_0 of the effective potential, i.e., $1-m\ll 1$. In these cases the above-barrier motion of an electron is strongly perturbed by the field of the standing wave and radiation can be emitted at frequencies ω_s which are multiples of the fundamental frequency ω_{res} found in Ref. 10.

The capabilities of the laser technology existing at that time were, however, inadequate for experimentally checking the results obtained in Ref. 10, and the frequencies calculated Ref. 10 were limited to the far-IR range. The electromagnetic radiation intensities that can now be achieved for picosecond and subpicosecond laser pulses with intensity $\sim 10^{16}$ W/cm^2 and higher at the focus make it possible to observe emission of near-IR and optical range photons by nonrelativistic electrons.

In the present paper, expressions for the harmonic frequencies and the corresponding intensities of the spontaneous radiation are obtained with the aid of the quasiclassical approximation for the case $m<1$ (the perturbation theory for the interaction of an electron with a standing wave) as well as for the case $1-m\ll 1$ (above-barrier motion of an electron). Estimates for the number of optical and near-IR photons emitted over one pulse from the region of the interaction of the electron beam with the standing wave are given for the parameters of powerful pulsed lasers with peak intensity $\sim 10^{16}$ W/cm^2. The possibilities of arranging the corresponding experiments are discussed.

2. FORMULATION OF THE PROBLEM. BASIC EQUATIONS

Let us consider the behavior of a nonrelativistic electron in the field of a linearly polarized electromagnetic standing wave. Let the field of the wave be given by the classical vector potential

$$\mathbf{A}_1(z,t)=2A_{01}\mathbf{e}_1\cos(\omega_1 t)\sin(k_1 z), \tag{1}$$

where A_{01} and $k_1(\omega_1,\pm\mathbf{k}_1)$ are, respectively, the amplitude and 4-momentum of waves which propagate in opposite directions along the z axis and form a standing wave, and $\mathbf{e}_1=\mathbf{e}_x$ is the unit polarization vector directed along the x axis.

We assume that the initial electron momentum $\mathbf{p}$ makes a small angle with the direction of the standing wave (the z axis), so that the longitudinal component of the momentum is much greater than the transverse component, $p_{\parallel}\gg p_{\perp}$. Generally speaking, an electron can be transmitted at an arbitrary angle relative to the direction of the wave. As will be shown below, however, the intensity of the harmonic radiation is proportional to the squared longitudinal length of the wave–electron interaction region, and for this reason the geometry in which the particles are directed along the wave is preferred.

In this geometry, since the ratio $e(\mathbf{A}_1\cdot\mathbf{p})/(eA_1)^2\ll 1$, the term $(eA_1)^2$ in the operator ($\hbar=c=1$)

$$\hat{V}=\frac{e(\mathbf{A}_1\cdot\mathbf{p})}{m_e}+\frac{(eA_1)^2}{2m_e}, \tag{2}$$

is responsible for the interaction of the electron with the wave.

The Schrödinger equation for a particle in the field of a standing wave has the form

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$$i\dot{\Psi}=-\frac{1}{2m_e}\Delta\Psi+V_0\sin^2(k_1z)\Psi, \tag{3}$$

where $V_0=(eA_{01})^2/m_e$ is the amplitude of the effective potential (Gaponov–Miller potential).

The solution of Eq. (3) is sought as a product of functions of the transverse coordinates x and y and the longitudinal coordinate z of the electron:

$$\Psi(\mathbf{r},t)=\exp[-i(\varepsilon t-\mathbf{p}_\perp\cdot\boldsymbol{\rho})]\psi(z), \tag{4}$$

where $\varepsilon=p^2/2m_e$ is the kinetic energy of the electron in the case when the field of the wave is switched off adiabatically and $\boldsymbol{\rho}(x,y)$ is the radius vector of the electron in the xy plane.

Substituting the expression (4) into Eq. (3) gives an equation for $\psi(z)$,

$$\varepsilon_\parallel\psi(z)=-\frac{1}{2m_e}\psi''(z)+V_0\sin^2(k_1z)\psi(z), \tag{5}$$

where $\varepsilon_\parallel=\varepsilon-\varepsilon_\perp$ and $\varepsilon_\perp=p_\perp^2/2m_e$ are the components of the electron energy associated with unperturbed motion of the electron parallel and transverse to the wave, and $\varepsilon_\parallel\approx\varepsilon$.

The solution of Eq. (5) obtained in the quasiclassical approximation has the form

$$\psi(z)=\exp\left[i\int\sqrt{2m_e[\varepsilon_\parallel-V(z)]}dz\right]. \tag{6}$$

The quasiclassicity condition $|\partial\lambda_{D\parallel}/\partial z|\ll 1$, where $\lambda_{D\parallel}$ is the de Broglie wavelength associated with the above-barrier motion of the electron along the wave, is given in terms of our problem by the inequality

$$\frac{m}{(1-m)^{3/2}}\frac{\omega_1}{(m_e\varepsilon_\parallel)^{1/2}}\ll 1,$$

and for initial electron energy $\varepsilon\gg\omega$, where ω is the frequency of the emitted photon, it imposes an upper limit on how close the parameter m can approach 1. Since the ratio $\omega_1/(m_e\varepsilon_\parallel)^{1/2}\sim 10^{-5}$ is small, the condition for quasiclassical electron motion is consistent with the inequality $1-m\ll 1$ (for $\varepsilon_\parallel\approx 10$ keV).

The field of the emitted wave with the vector potential

$$\mathbf{A}_2=(A_{02}/2)[\mathbf{e}\exp[i(\omega t-\mathbf{k}\cdot\mathbf{r})]+\text{c.c.}] \tag{7}$$

(A_{02} and $\mathbf{e}$ are, respectively, the amplitude and polarization unit vector) is assumed to be weak and is taken into account by means of perturbation theory.

To first order in the field (7) the amplitude for the transition of an electron from the initial state $(\varepsilon,\mathbf{p})$ into the final state $(\varepsilon',\mathbf{p}')$ with the emission of a photon $(\omega,\mathbf{k})$ is determined by the expression

$$A_{fi}(t)=-i\frac{eA_{02}}{2m_eV}\int_0^t\int\exp[i(\varepsilon'+\omega-\varepsilon)t_1 -(\mathbf{p}'_\perp+\mathbf{k}_\perp-\mathbf{p}_\perp)\cdot\boldsymbol{\rho}]\psi_f^*(z)\left(\mathbf{e}_\perp\cdot\mathbf{p}_\perp-ie_z\frac{\partial}{\partial z}\right) \times\exp(-ik_zz)\psi_i(z)d\boldsymbol{\rho}\,dz\,dt_1, \tag{8}$$

where $\psi_i(z)$ and $\psi_f(z)$ are, respectively, the initial and final wave functions (6) and V is the normalization volume of the particle.

Assuming that the relative change in the longitudinal component of the electron momentum during the emission process is small, $|\Delta p_\parallel/p_\parallel|\sim\omega/[(1-m)\varepsilon_\parallel]\ll 1$, after integrating in Eq. (8) over the time and the transverse coordinates we obtain

$$A_{fi}(t)=-i\frac{eA_{02}}{2m_ek_1}\frac{\exp[i(\varepsilon'+\omega-\varepsilon)t]-1}{i(\varepsilon'+\omega-\varepsilon)}\left[\frac{2J_1(u)}{u}\right]\frac{1}{l} \times\left\{(\mathbf{e}_\perp\cdot\mathbf{p}_\perp)\int_{-\pi N/2}^{\pi N/2}\exp[-i\Delta[F(\xi|m) +(k_z/\Delta p_\parallel)\xi]]d\xi+(e_zp_\parallel)\int_{-\pi N/2}^{\pi N/2}(1 -m\sin^2\xi)^{1/2}\exp[-i\Delta[F(\xi|m) +(k_z/\Delta p_\parallel)\xi]]d\xi\right\}. \tag{9}$$

In Eq. (9) we introduced the notation

$$u=|\Delta\mathbf{p}_\perp+\mathbf{k}_\perp|\rho_0\sin\theta, \tag{10}$$

where θ is the angle made by the vector $\Delta\mathbf{p}+\mathbf{k}$ with the z axis; $\Delta p_\parallel=p'_\parallel-p_\parallel$ and $\Delta\mathbf{p}_\perp=\mathbf{p}'_\perp-\mathbf{p}_\perp$ are, respectively, the changes in the longitudinal and transverse components of the electron momentum; $\Delta\equiv\Delta p_\parallel/k_\parallel$; $N=2l/\lambda_1$ is the number of standing waves which fit within the electron–wave interaction region; $l=v_\parallel\tau$ is the distance traversed by an electron along the standing wave during the pulse time τ; $\lambda_1=2\pi/k_1$;

$$F(\xi|m)=\int_0^\xi(1-m\sin^2x)^{-1/2}dx$$

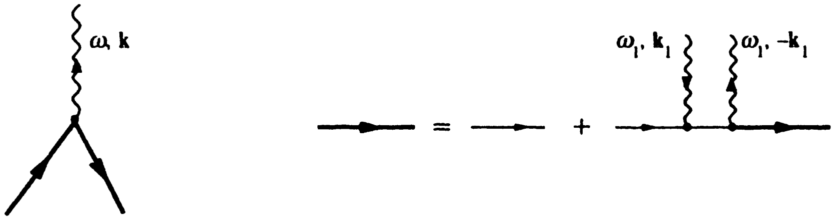
is an elliptic integral of the first kind;[11] and, $J_1(u)$ is a Bessel function.

In Eq. (9) it is assumed that $V=\pi\rho_0^2l$, where ρ_0 is the radius of the focus at the center of the standing wave (in the plane $z=0$).

Squaring the expression (9) and using the well-known representations for a delta function ($\omega_1\tau\gg 1$, $\rho_0\gg\lambda$, where λ is the spontaneous-emission wavelength), we obtain for the probability of a transition per unit time into the partial final state

$$\frac{|A_{fi}(t)|^2}{t}=\left(\frac{eA_{01}}{2m_e}\right)^2\frac{2\lambda_1^2}{\rho_0^2}\delta(\varepsilon'+\omega-\varepsilon) \times\delta^{(2)}(\Delta\mathbf{p}_\perp+\mathbf{k}_\perp)\frac{1}{l^2}|(\mathbf{e}_\perp\cdot\mathbf{p}_\perp)I_1 +(e_zp_\parallel)I_2|^2, \tag{11}$$

where



a b FIG. 1.

$$I_1=\int_{-\pi N/2}^{\pi N/2}\exp\{i[|\Delta|F(\xi|m)-\beta\xi]\}d\xi,$$

$$I_2=\int_{-\pi N/2}^{\pi N/2}(I-m\,\sin^2\,\xi)^{1/2}$$

$$\times\exp\{i[|\Delta|F(\xi|m)-\beta\xi]\}d\xi,$$

$$\beta\equiv k_z/k_1. \tag{12}$$

Integrating over the transverse momentum of the scattered electron, we obtain from Eq. (11) the following expression for the spectral–angular density of the intensity of the spontaneous radiation:

$$\frac{d^2I}{d\omega\,d\Omega_{\mathbf{k}}}=\frac{\alpha\lambda_1^2\omega^2}{\pi^2(2m_e)^2}\,\delta(\varepsilon'+\omega-\varepsilon)\,\frac{1}{l}\,|(\mathbf{e}_\perp\cdot\mathbf{p}_\perp)I_1$$

$$+(e_zp_\parallel)I_2|^2dp_\parallel', \tag{13}$$

where $\alpha=e^2/\hbar c$ is the fine structure constant.

From the law of conservation of the transverse component of the momentum of the system $\mathbf{p}_\perp'+\mathbf{k}_\perp-\mathbf{p}_\perp=0$ it is easy to express the final energy ε' of the electron in terms of the parameters of its initial state and the characteristics of the emitted photon:

$$\varepsilon'=(p_\parallel'^2+p_\perp'^2)/2m_e=[p_\perp^2-2(\mathbf{p}_\perp\cdot\mathbf{k}_\perp)+k_\perp^2$$

$$+p_\parallel'^2]/2m_e. \tag{14}$$

Using a well-known property of delta functions, with the aid of Eq. (14) we represent $\delta(\varepsilon'+\omega-\varepsilon)$ in the form

$$\delta(\varepsilon'+\omega-\varepsilon)=\frac{m_e}{p_0}\,[\delta(p_\parallel'-p_0)+\delta(p_\parallel'+p_0)], \tag{15}$$

where

$$p_0=(p_\parallel^2+2(\mathbf{p}_\perp\cdot\mathbf{k}_\perp)-2m_e\omega-k_\perp^2)^{1/2}. \tag{16}$$

Substituting the expression (15) into Eq. (13) and integrating over $dp_\parallel'$, we obtain

$$\frac{d^2I}{d\omega\,d\Omega_{\mathbf{k}}}=\frac{2\alpha\lambda_1^2\omega^2}{(2\pi)^4(2m_e)^{3/2}\varepsilon^{1/2}}\,\frac{1}{l}\,|(\mathbf{e}_\perp\cdot\mathbf{p}_\perp)I_1$$

$$+(e_zp_\parallel)I_2|^2, \tag{17}$$

where the approximate equality $p_0\approx p_\parallel\approx(2m_e\varepsilon)^{1/2}$ was used in the denominator.

The expression (17) is general, and further calculations depend on the value taken for the parameter m. We shall study two limiting cases: small $m<1$, when the problem is solved by perturbation theory, and above-barrier motion of an electron, when the parameter m is close to 1, so that $1-m\ll 1$. In the latter case perturbation theory is inapplicable, and the problem must be solved exactly without using an expansion of the transition amplitude in a series in the small parameter m.

In closing this section we give a graphical interpretation of the effect with the aid of Feynman diagrams. A diagram of the spontaneous emission of a low-frequency photon $(\omega,\mathbf{k})$ in the field of a standing wave is displayed in Fig. 1a. The thick electron lines correspond to a state of the electron in the field of a strong standing wave. These states originate from multiple rescattering of the wave photons forming the standing wave by the electron. This process is represented graphically in Fig. 1b.

3. THE CASE OF HIGH ELECTRON ENERGIES (PERTURBATION THEORY)

Assuming m to be small, we expand the function $F(\xi|m)$ in a series in m and retain the lowest-order terms:

$$F(\xi|m)\approx\left(1+\frac{m}{4}\right)\xi-\frac{m}{8}\left(1+\frac{3}{4}\,m\right)\sin 2\xi \tag{18}$$

(the terms proportional to higher powers of m are small for numerical reasons and therefore are dropped in Eq. (18)).

Substituting the expansion (18) into Eq. (12), we obtain for the integrals I_1 and I_2

$$I_1=\int_{-\pi N/2}^{\pi N/2}\exp\left\{i\left[\left(1+\frac{m}{4}\right)\left|\Delta\right|-\beta\right]\xi\right\}$$

$$\times\exp\left[-i\left|\Delta\right|\frac{m}{8}\left(1+\frac{3}{4}\,m\right)\sin 2\xi\right]d\xi,$$

$$I_2=\int_{-\pi N/2}^{\pi N/2}\left(1-\frac{m}{2}\,\sin^2\,\xi\right)\exp\left\{i\left[\left(1+\frac{m}{4}\right)\left|\Delta\right|-\beta\right]\xi\right\}$$

$$\times\exp\left(-i\left|\Delta\right|\frac{m}{8}\left(1+\frac{3}{4}\,m\right)\sin 2\xi\right)d\xi. \tag{19}$$

The integrals (19) are easily calculated with the aid of the well-known expansion[12]

$$\exp\left[-i\left|\Delta\right|\frac{m}{8}\left(1+\frac{3}{4}\,m\right)\sin 2\xi\right]$$

$$=\sum_{s=1}^{\infty}J_s\left[\left|\Delta\right|\frac{m}{8}\left(1+\frac{3}{4}\,m\right)\right]\exp(-is2\xi). \tag{20}$$

Substituting the expression (20) into Eq. (19) and integrating over ξ, it follows from Eq. (17) that

$$\frac{d^2 I}{d\omega\, d\Omega_{\mathbf{k}}} = \frac{\alpha l}{2\pi^2}\left(\frac{\varepsilon}{2m_e}\right)^{1/2} (\mathbf{e}\cdot\mathbf{n})^2 \sum_{s=1}^{\infty} (\omega_s)^2 J_s^2 \times\left[\left|\Delta\right| \frac{m}{8}\left(1+\frac{3}{4}m\right)\right] \frac{\sin^2 u_s}{u_s^2}, \tag{21}$$

where $\mathbf{n}=\mathbf{p}/p$ is a unit vector in the direction of the initial electron momentum and the factor u_s is given by the formula

$$u_s = \left[\left(1+\frac{m}{4}\right)\left|\Delta\right| - 2s - \beta\right] \frac{\pi N}{2} \approx (|\Delta| - 2s)\,\frac{\pi N}{2} \tag{22}$$

(in the final expression for u_s, small terms $m/4$ and β, which have virtually no effect on the radiation frequencies, are dropped).

The expression (21) was obtained under the assumption that the spectral emission lines of the electron which correspond to different harmonics s do not overlap. As will be shown below, this type of spectrum arises if the homogeneous width of the lines is sufficiently small, which presumes that the admissible duration of the laser pulse satisfies a definite condition. Furthermore, the transition from Eq. (17) to Eq. (21) was made assuming that $I_2 \approx I_1$. When the differences of these integrals are taken into account, small corrections $\sim (m/4)^2$ appear in the expression for $d^2 I / d\omega d\Omega_{\mathbf{k}}$.

Integrating in Eq. (21) over all directions for the emergence of a photon it is easy to obtain an expression for the spectral density of the intensity of the spontaneous radiation from an electron in the field of a standing wave (under the condition $m<1$)

$$\frac{dI}{d\omega} = \frac{4}{3\pi}\,\alpha\left(\frac{\varepsilon}{2m_e}\right)^{1/2} l \sum_{s=1}^{\infty} \omega_s^2 J_s^2(s\tilde{m}/4)\,\frac{\sin^2 u_s}{u_s^2}, \tag{23}$$

where $\tilde{m} = m(1+m/2)$.

It is easy to see from Eq. (23) that the position of the lines is determined by the diffraction factor $\sin^2 u_s / u_s^2$, and the envelope of the intensity of individual harmonics is described mainly by the function $J_s^2(s\tilde{m}/4)$.

An expression for the characteristic radiation frequencies follows from the equality $u_s = 0$:

$$\omega_s = s 4\omega_1 \left(\frac{\varepsilon}{2m_e}\right)^{1/2} \tag{24}$$

(we call attention to the fact that the formula (24) is approximate: The term $k_\perp^2 / 2m_e\omega$, due to the electron recoil accompanying the emission of a photon, as well as the small terms $\mathbf{p}_\perp \cdot \mathbf{k}_\perp / m_e \omega \sim v_\perp / c \ll 1$ and $k_z / |\Delta p_\parallel| \sim v_\parallel / c \ll 1$ are dropped).

As follows from Eq. (24), the emission lines are equidistant with spacing ω_{res}, where

$$\omega_{\text{res}} = 4\omega_1 \left(\frac{\varepsilon}{2m_e}\right)^{1/2} \tag{25}$$

is the fundamental resonance frequency, determined by the transit time of an electron between two neighboring maxima of the ponderomotive potential.

The frequency profile of the s-th harmonic is given by the expression

$$\frac{dI_s}{d\omega} = \frac{4}{3\pi}\,\alpha\left(\frac{\varepsilon}{2m_e}\right)^{1/2} l\,\omega_s^2 J_s^2(s\tilde{m}/4) \times \frac{\sin^2[(\omega/\omega_{\text{res}} - s)2\pi l/\lambda_1]}{[(\omega/\omega_{\text{res}} - s)2\pi l/\lambda_1]^2}, \tag{26}$$

which follows from Eq. (23).

The width of the emission lines, which is determined from the condition $|u_s| \sim \pi$, equals

$$\delta\omega_h \approx \omega_{\text{res}} \lambda_1 / l = 4\pi/\tau. \tag{27}$$

The expression (27) is the homogeneous width of the lines and relates the width to the finiteness of the effective longitudinal electron–wave interaction length. In the present geometry, the quantity l depends on the laser pulse duration τ and the electron energy.

The condition that the spectral emission lines not overlap with one another is given by the obvious inequality $\delta\omega_h < \omega_{\text{res}}$ and presumes a pulse duration such that there is enough time for an electron to traverse a large number of wavelengths of the standing wave during the lifetime of the field. For fixed electron energy $\varepsilon = V_0/m$ this inequality gives a lower limit for the minimum pulse duration:

$$\tau > \frac{\lambda_1}{2}(2m_e/\varepsilon)^{1/2}. \tag{28}$$

For the numerical values taken in the present paper for the main parameters of the problem (see below), the duration satisfying the condition (28) is $\tau \approx 10$ fs (for $\varepsilon \approx 37$ keV and $\lambda_1 = 1\cdot 10^{-4}$ cm). The same criterion (28) makes it possible to use the Gaponov–Miller potential in the interaction operator in the initial equation (3).

Under certain conditions, the dependence of the frequency ω_{res} (25) on the electron energy can result in a strong inhomogeneous broadening of the lines and, in consequence, the peaks of different harmonics of the fundamental resonance frequency can overlap. The corresponding inhomogeneous width is given by the formula

$$\delta\omega_{nh} \approx \frac{1}{2} s\,\omega_{\text{res}} \left(\frac{\delta\varepsilon}{\varepsilon}\right). \tag{29}$$

A characteristic feature of the expression (29) is that the inhomogeneous width, in contrast to $\delta\omega_h$ (27), increases linearly with s. From the expressions (27) and (29) arises a condition for the degree of beam nonmonochromaticity up to which the width $\delta\omega_h$ ($\delta\omega_{nh} < \delta\omega_h$) predominates:

$$\frac{\delta\varepsilon}{\varepsilon} < \frac{2\lambda_1}{sl} = \frac{\lambda_1}{s\tau}\left(\frac{2m_e}{\varepsilon}\right)^{1/2}. \tag{30}$$

For emission at the fundamental frequency ($s=1$) and pulse duration $\tau = 0.1$ ps with $\varepsilon = 37$ keV and $\lambda_1 = 1\cdot 10^{-4}$ cm, the inequality (30) holds with a weak condition on beam monochromaticity: $\delta\varepsilon/\varepsilon < 0.18$.

Integrating the expression (26) over frequency gives the total intensity of the radiation of the sth harmonic:

$$I_s=\frac{(32\pi)^2}{3}\alpha\left(\frac{\varepsilon}{2m_e}\right)^2\frac{1}{\lambda_1^2}s^2J_s^2(s\tilde{m}/4), \tag{31}$$

whose value is thus related with the homogeneous width $\delta\omega_h$ (27).

It is easy to obtain from Eq. (31) an expression, which is convenient for comparing with experiment, for the number of photons of the s-th harmonic which are emitted from the electron–wave interaction volume by the electron beam over the lifetime of the standing wave (in ordinary units):

$$N_s=\frac{2(8\pi)^2}{3}\alpha\left(\frac{\varepsilon}{2m_ec^2}\right)^{3/2}\frac{n_e\rho_0^2(c\tau)^2}{\lambda_1}sJ_s^2(s\tilde{m}/4), \tag{32}$$

where n_e is the electron density in the beam.

Since the parameter m is small, the formula (32) can be simplified with the aid of the well-known series expansion of the Bessel function with respect to a small argument:[11]

$$J_\nu(z)\sim(2\pi\nu)^{-1/2}\left(\frac{ez}{2\nu}\right)^\nu .$$

As a result, we obtain from Eq. (32)

$$N_s=\frac{64\pi}{3}\alpha\left(\frac{\varepsilon}{2m_ec^2}\right)^{3/2}\frac{n_e\rho_0^2(c\tau)^2}{\lambda_1}\left(\frac{e\tilde{m}}{8}\right)^{2s}. \tag{33}$$

The formula (33) expresses a natural result for perturbation theory: The number N_s is a power-law function of the amplitude of the ponderomotive potential ($N_s\propto V_0^{2s}$). Since m is small, it follows from Eq. (33) that the number of photons emitted per pulse drops rapidly with increasing harmonic number s:

$$N_s\propto(e\tilde{m}/8)^{2s}.$$

4. THE CASE OF ABOVE-BARRIER MOTION OF AN ELECTRON

We return to the general formula (17). To calculate the integrals I_1 and I_2 we divide the region of integration $(-\pi N/2,\ \pi N/2)$ into successive sections of length π. Using the periodicity of the integrand in $F(\xi|m)$ expressed by the equation[11]

$$F(s\pi\pm\xi|m)=2sK(m)\pm F(\xi|m), \tag{34}$$

where

$$K(m)=\int_0^{\pi/2}(1-m\sin^2x)^{-1/2}dx$$

is a complete elliptic integral of the first kind, we can represent the absolute value in Eq. (17) in the form

$$|\ |=\frac{\exp[i[2|\Delta|K(m)-\beta\pi]N/2]-\exp[-i[2|\Delta|K(m)-\beta\pi]N/2]}{\exp[i[2|\Delta|K(m)-\beta\pi]]-1}[(\mathbf{e}_\perp\cdot\mathbf{p}_\perp)\tilde{I}_1+(e_zp_\parallel)\tilde{I}_2], \tag{35}$$

where $\tilde{I}_1$ and $\tilde{I}_2$ are the integrals

$$\tilde{I}_1=\int_0^\pi\exp\{i[|\Delta|F(\xi|m)-\beta\xi]\}d\xi,$$

$$\tilde{I}_2=\int_0^\pi(1-m\sin^2\xi)^{1/2}\exp\{i[|\Delta|F(\xi|m)-\beta\xi]\}d\xi. \tag{36}$$

Without repeating the computations performed in Sec. 3, using Eq. (35) we obtain from the formula (17) an expression for the spectral-angular density of the intensity of the spontaneous radiation:

$$\frac{d^2I}{d\omega\, d\Omega_{\mathbf{k}}}=\frac{\alpha\lambda_1^2\omega^2}{(2\pi)^4m_ep_\parallel}\frac{\sin^2(N\delta/2)}{\sin^2(\delta/2)}\frac{1}{l}\times|(\mathbf{e}_\perp\cdot\mathbf{p}_\perp)\tilde{I}_1+(e_zp_\parallel)\tilde{I}_2|^2, \tag{37}$$

where the approximation $p_0\approx p_\parallel$ was used in the denominator and $\delta\equiv2|\Delta|K(m)-\beta\pi\approx2|\Delta|K(m)$.

The integrals $\tilde{I}_1$ and $\tilde{I}_2$ are calculated in the Appendix. Using the results obtained there (Eq. (A3)), we give the following expression for the spectral density of the spontaneous radiation intensity:

$$\frac{dI}{d\omega}=\left(\frac{8}{3}\right)^2\alpha\left(\frac{\varepsilon}{2m_e}\right)^{1/2}\frac{\lambda_1^2\omega^2}{(2\pi)^3}\frac{1}{l}\frac{\sin^2(2\pi(l/\lambda_1)(\omega/\tilde{\omega}_{\rm res}))}{\sin^2(\pi\omega/\tilde{\omega}_{\rm res})}\times\frac{1}{m}K_{1/3}^2\left(\frac{\sqrt{2}}{3}\frac{2\pi}{m^{1/2}K(m)}\frac{\omega}{\tilde{\omega}_{\rm res}}\right), \tag{38}$$

where $K_{1/3}(v)$ is a modified Bessel function[12] and a formula for $\tilde{\omega}_{\rm res}$ is presented below.

Comparing the expression (38) with Eqs. (23) and (26) shows that the case of above-barrier motion of an electron reproduces the main features of the radiation spectrum which we studied in Sec. 3 (the case of small m). For example, the resonance condition $\delta/2=s\pi$, where $s=1,\ 2,\dots$, arising from the diffraction factor $\sin^2(N\delta/2)/\sin^2(\delta/2)$ in Eq. (38) determines the expression for the characteristic radiation frequencies

$$\tilde{\omega}_s=s\omega_1\frac{2\pi}{K(m)}\left(\frac{\varepsilon}{2m_e}\right)^{1/2}. \tag{39}$$

A consequence of Eq. (39) is that the lines are equidistant with spacing $\tilde{\omega}_{\rm res}$, where

$$\tilde{\omega}_{\rm res}=\omega_1\frac{2\pi}{K(m)}\left(\frac{\varepsilon}{2m_e}\right)^{1/2} \tag{40}$$

is the fundamental resonant frequency, which now depends not only on the electron energy but also on the value of the parameter $m=V_0/\varepsilon$ (compare with Eq. (25)).

The envelope of the peaks in the emission spectrum is given by the function $K_{1/3}^2(v)$ in the formula (38). In the case when for numerical reasons the argument of the modified Bessel function is greater than 1

$$v_s=\frac{\sqrt{2}}{3}\frac{2\pi}{m^{1/2}K(m)}s>1$$

(for the parameters of the problem that can be realized experimentally this condition holds even for $s=1$), the asymptotic expansion of the function $K_{1/3}$ can be used.[12] Then, as follows from Eq. (38), the maximum of the spectral density for the peak with number s is determined mainly by the relation

$$\frac{dI_s}{d\omega}\propto \exp\left(-\frac{2^{3/2}}{3}\frac{2\pi}{m^{1/2}K(m)}s\right). \tag{41}$$

The expression (41) indicates that, as in the case of small m, the intensity of the harmonic drops rapidly with increasing harmonic number s. Now, however, this dependence is exponential and not a power law.

The homogeneous width of the emission lines is determined by the diffraction factor in Eq. (38), and for fixed ε and m is given by the expression

$$\delta\tilde{\omega}_h=\frac{\tilde{\omega}_{\rm res}\lambda_1}{l}=\frac{2\pi^2}{\tau K(m)}. \tag{42}$$

Just as in the case when perturbation theory is applicable, the lines are resolved when the inequality (28) holds.

We shall show that the dependence of the frequency $\tilde{\omega}_{\rm res}$ (40) on the parameters ε and m can result in strong inhomogeneous line broadening. Now, when $1-m\ll 1$ holds, there exist two independent sources of this broadening. One source is due, as in the case of small m, to the initial energy spread of the electrons in the beam. The corresponding inhomogeneous width equals, as follows from Eq. (39),

$$\delta\tilde{\omega}_{nh}=\frac{1}{2}s\tilde{\omega}_{\rm res}\left[1+\frac{2mK'(m)}{K(m)}\right]\frac{\delta\varepsilon}{\varepsilon}, \tag{43}$$

where $K'(m)=dK(m)/dm$ and $\delta\varepsilon$ is the width of the distribution of the initial energy of the electrons. It is easy to see that in the limit of small m the formula (43) goes over to Eq. (29). The additional term $2mK'(m)/K(m)$ in the square brackets in Eq. (43) arises as a result of the dependence of the frequency $\tilde{\omega}_{\rm res}$ on the parameter m. To simplify the expression (43), we employ the well-known approximate (in the limit $m\to 1$) formula for the complete elliptic integral[12]

$$K(m)\approx\frac{1}{2}\ln\left(\frac{16}{1-m}\right).$$

Its derivative is $K'(m)=1/2(1-m)$. For the value $m=0.95$ employed in the estimates, we have $2mK'(m)/K(m)\approx 7$ and therefore this factor makes the main contribution to the inhomogeneous width:

$$\delta\tilde{\omega}_{nh}\approx s\tilde{\omega}_{\rm res}\frac{mK'(m)}{K(m)}\frac{\delta\varepsilon}{\varepsilon}. \tag{44}$$

Note that the width $\delta\tilde{\omega}_{nh}$ increases as $m\to 1$.

The condition under which the homogeneous width of the lines plays the dominant role ($\delta\tilde{\omega}_{nh}<\delta\tilde{\omega}_h$) is expressed by the inequality

$$\frac{\delta\varepsilon}{\varepsilon}<\frac{\lambda_1}{2s\tau}\left(\frac{2m_e}{\varepsilon}\right)\frac{1}{mK'(m)/K(m)},$$

which under otherwise the same conditions imposes a more stringent restriction on the degree of beam nonmonochromaticity.

The second source of inhomogeneous line broadening is the actual inhomogeneity of the field of the standing wave in the transverse directions. To obtain the estimates below, we describe the function $V_0(\rho)$ by a simple Gaussian distribution

$$V_0(\rho)=V_0\exp(-\rho^2/\rho_0^2) \tag{45}$$

(for laser pulse duration $\tau\leqslant 1$ ps the longitudinal extent of the standing wave is much smaller than the confocal parameter, and in Eq. (45) the dependence of the radius ρ_0 of the focus on the longitudinal coordinate z is dropped).

The line width due to the spatial dependence of the ponderomotive potential V_0 on the coordinate ρ is determined by the formula

$$\delta\tilde{\omega}'_{nh}\approx s\tilde{\omega}_{\rm res}\frac{mK'(m)}{K(m)}\frac{2\rho}{\rho_0^2}\rho_c, \tag{46}$$

which follows from Eqs. (39) and (45) with fixed ε. If the factor $K_{1/3}^2$, which gives the envelope of the spectral density, were not present in the formula (38), then $\rho\approx\rho_0/2$ and $\rho_c\approx\rho_0$ would have to be used for ρ and ρ_0 in order to estimate the line width. The inhomogeneous width arising with this choice of the parameters in Eq. (46) would be so large that it would exceed the spacing $\tilde{\omega}_{res}$ between separate lines.

In reality the factor $K_{1/3}^2$ in Eq. (38) plays the determining role in the estimate of $\delta\tilde{\omega}'_{nh}$ from the formula (46). This is because the argument of the modified Bessel function contains the parameter m, whose dependence on ρ is reproduced by the formula (45). When the distance ρ increases as the radiating electron moves away from the axis of the focus, m decreases and, in consequence, the radiation intensity decreases exponentially (see Eq. (41)). It is easy to estimate from Eq. (41) the radius of the region of the standing-wave field near the axis that makes the main contribution to the intensity of the radiation of the sth harmonic:

$$\tilde{\rho}_0\approx\rho_0\left[\frac{3}{2^{5/2}\pi s m^{1/2}K'(m)}\right]^{1/2}K(m). \tag{47}$$

Setting $\rho\approx\tilde{\rho}_0/2$ and $\rho_c\approx\tilde{\rho}_0$, we obtain from Eq. (46) the following expression for the inhomogeneous width:

$$\delta\tilde{\omega}'_{nh}\approx\omega_{\rm res}\frac{3m^{1/2}K(m)}{2^{5/2}\pi}, \tag{48}$$

which, in contrast to Eq. (44), does not depend on the harmonic number.

The estimates presented below show that for the values taken for the main parameters of the problem the inhomogeneous width $\delta\tilde{\omega}'_{nh}$ plays the dominant role and exceeds the homogeneous width by approximately an order of magnitude, reaching values $\approx(1/2)\tilde{\omega}_{\rm res}$. Therefore, in contrast to the case when perturbation theory is applicable $(m<1)$, under the conditions of above-barrier motion of the electrons the nonuniformity of the focus of the standing wave in the transverse directions strongly influences the character of the spectrum.

It is easy to obtain from Eq. (38) an expression for the number of photons of the sth harmonic which are emitted by the electron beam over the lifetime of the standing wave:

$$\tilde{N}_s=\left(\frac{8}{3}\right)^2 3\sqrt{2}\pi\alpha\left(\frac{\varepsilon}{2m_e}\right)^{3/2}\frac{n_e\tilde{\rho}_0^2(c\tau)^3}{\lambda_1^2}\times\frac{\sqrt{m}}{K(m)}sK_{1/3}^2\left(\frac{\sqrt{2}}{3}\frac{2\pi}{m^{1/2}K(m)}s\right). \quad (49)$$

As follows from Eqs. (33) and (49), a comparative estimate of the number of photons emitted by electrons in the regime of above-barrier motion or under the conditions when perturbation theory is applicable is given by the ratio

$$\frac{\tilde{N}_s}{N_s}=\frac{27}{2(2\pi)^2}\frac{c\tau}{\lambda_1}\frac{K(m)}{K'(m)}s^{-1}\frac{K_{1/3}^2(2^{3/2}\pi/3m^{1/2}K(m))}{J_s^2(s\tilde{m}/4)}. \quad (50)$$

Substituting into Eq. (50) the parameters adopted for the main quantities (see below) shows that for radiation at the fundamental harmonic $(s=1)$ the number $\tilde{N}_s$ is approximately an order of magnitude larger than the number N_s.

5. NUMERICAL RESULTS AND CONCLUSIONS

In this section we shall present the results of numerical calculations, performed according to the formulas obtained above, for the harmonic frequencies and the corresponding intensities of the spontaneous radiation. We shall employ below the following parameters for the laser radiation: $\lambda_1=800$ nm ($\omega_1=1.55$ eV), $\tau=100$ fs (data for a titanium-sapphire laser) and focal radius $\rho_0=1\cdot10^{-2}$ cm. For pulse energy $W=0.5$ J the peak intensity at the center of the focus is $I=1.6\cdot10^{16}$ W/cm^2 and the amplitude of the ponderomotive potential is $V_0=3.7$ keV.

For the case of small m we take for the parameters of the electron beam $m=0.1$ (electron energy $\varepsilon=37$ keV), beam diameter $d_e=0.1$ cm, $n_e=1\cdot10^{12}$ cm^{-3}, and $\delta\varepsilon/\varepsilon=0.1$.

The fundamental frequency calculated according to the formula (25) reaches the limit of the optical range and equals $\omega_{res}=1.2$ eV. The number of photons emitted per pulse by the electron beam at the fundamental frequency $(s=1)$ equals $N_{s=1}\approx60$. For the adopted values of the parameters the number of photons in the second harmonic $(s=2)$ with frequency $\omega=2.4$ eV is much smaller and equals $N_{s=2}\approx0.1$.

We give a parametric relation for the number of photons emitted at the fundamental frequency (see Eq. (33)):

$$N_s\propto\frac{1}{\sqrt{\varepsilon}}\frac{W^2}{\rho_0^2}\lambda_1^3 n_e.$$

For the case of above-barrier motion of electrons we take at the center of the focal spot $m=0.95$ (the electron energy $\varepsilon=3.9$ keV). The radius of the near-axis region of the standing-wave field is, according to Eq. (47), $\tilde{\rho}_0\approx0.4\rho_0$. According to Eq. (49), the fundamental frequency of the radiation $(s=1)$ is $\tilde{\omega}=0.21$ eV. For $\delta\varepsilon/\varepsilon=0.1$ the inhomogeneous line width due to beam nonmonochromaticity reaches the value $\delta\tilde{\omega}_{nh}\approx0.07$ eV. The inhomogeneous width $\delta\tilde{\omega}'_{nh}$ (48) is $\delta\tilde{\omega}'_{nh}\approx0.1$ eV and plays the dominant role.

The regime of electron motion under study is characterized by lower frequencies but much higher intensities of the spontaneous radiation in the lowest-order harmonics. For example, the number of photons emitted over a pulse at the fundamental frequency by electrons from the near-axis region of the focus is $\tilde{N}_{s=1}\approx7\cdot10^2$. The number of photons in the second harmonic with frequency $\omega=0.42$ eV is also quite large and equals $\tilde{N}_{s=2}\approx1\cdot10^2$. An estimate according to the formula (49) gives for the third harmonic $\tilde{N}_{s=3}\approx1$.

It is evident from the estimates presented that two types of experiments can be performed. In one type the electron energy is much higher than the amplitude of the ponderomotive potential $(m\approx0.1)$. Under these conditions the dependence of the potential V_0 on the transverse coordinate ρ is very weak, and practically all electrons in the volume of the focus participate in photon emission. The emission spectrum is actually represented by a single line corresponding to the fundamental frequency $\omega_{\rm res}$. Under the condition (30) the line profile is given by the homogeneous width (27) and depends on the duration of the pulse.

In experiments of the other type the electron energy is close to the amplitude of the ponderomotive potential at the center of the focus $(1-m\ll1)$. Under these conditions the dependence of the potential V_0 on the transverse coordinate ρ is substantial and results in substantial inhomogeneous broadening of the emission lines. Several lines corresponding to the lowest-order harmonics have an appreciable intensity in the emission spectrum. The frequencies of these harmonics are comparatively low and lie in the near-IR range.

It should be noted that the radiation by an electron in the field of a standing wave can occur not only as a result of the interaction $\sim\mathbf{A}_2\cdot\mathbf{p}$, which we studied, but also as a result of the term $\sim\mathbf{A}_1\cdot\mathbf{A}_2$ in the perturbation operator. The relative role of these terms is determined by the parameter

$$\eta\sim\frac{e^2\mathbf{A}_1\cdot\mathbf{A}_2}{e\mathbf{A}_2\cdot\mathbf{p}}\approx\frac{eE_1\lambda_1}{\sqrt{2m_e\varepsilon}},$$

where the electron energy must exceed the amplitude of the ponderomotive potential. The maximum value of the parameter η corresponds to the case $\varepsilon\geqslant V_0$. In this case it satisfies $\eta\approx1/\sqrt{2}$ and does not depend on the intensity of the pump wave. In the case when perturbation theory is applicable, $\varepsilon\gg V_0$, the parameter satisfies $\eta\ll1$.

Furthermore, the spectrum and angular distribution of the spontaneous radiation due to the interaction $\sim\mathbf{A}_1\cdot\mathbf{A}_2$ differ substantially from the low-frequency radiation studied in the present paper. Specifically, the spectrum of this radiation consists of lines whose frequencies are equal to or are multiples of the frequency ω_1 of the standing wave, and in contrast to the radiation studied above it is emitted along the axis of the standing wave. Nonetheless, an investigation of the radiation due to the interaction $\sim \mathbf{A}_1 \cdot \mathbf{A}_2$ is of interest and is the subject of a separate analysis.

In closing, we thank Dr. P. Agostini for a discussion of the possibility of carrying out an experiment.

6. APPENDIX

We shall calculate the integral $\widetilde{I}_1$ (32). The integral $\widetilde{I}_2$ is calculated completely analogously and, as will be seen from the subsequent calculations, equals $\widetilde{I}_1$ for $|\Delta| \gg 1$. Let us divide the region of integration $(0,\pi)$ into two sections $(0,\pi/2)$ and $(\pi/2,\pi)$. Making the substitution of variables $\xi = \pi + \xi'$ in the second integral (integration over the section $(\pi/2,\pi)$) and using the equality (3) and the antisymmetry of the function $F(\xi|m)$, we obtain

$$\widetilde{I}_1 = \int_0^{\pi/2} \exp\{i[|\Delta| F(\xi|m) - \beta\xi]\} d\xi + e^{i\delta} \times \int_0^{\pi/2} \exp\{-i[|\Delta| F(\xi|m) - \beta\xi]\} d\xi = 2\int_0^{\pi/2} \cos[|\Delta| F(\xi|m) - \beta\xi] d\xi \approx 2\int_0^{\pi/2} \cos[|\Delta| F(\xi|m)] d\xi \tag{A1}$$

(the term containing β in the phase of the integrand is relatively small ($\sim 1/|\Delta|$) and does not affect the value of the integral). According to the resonance condition, $\delta = 2s\pi$, where $s = 1,2,\ldots$, so that the $\exp(i\delta) = 1$.

The function $F(\xi|m)$ grows monotonically with ξ, so that for $|\Delta| \gg 1$ small values of ξ play the main role in the integral (A1) (for values of ξ which are not small the integrand oscillates rapidly). Expanding $F(\xi|m)$ in powers of ξ around the point $\xi = 0$ and retaining the lowest powers of ξ and the parameter m we find

$$F(\xi|m) \approx \xi + (m/6)\xi^3 \quad (\text{for } \xi \ll 1). \tag{A2}$$

Since the section of integration where $\xi_0 \sim 1/|\Delta| \ll 1$ makes the main contribution to the value of the integral (A1), replacing the upper limit of integration by ∞ and substituting the expression (A2) into Eq. (A1) we obtain

$$\widetilde{I}_1 \approx 2\int_0^{\infty} \cos\{|\Delta|[\xi + (m/6)\xi^3]\} d\xi.$$

This expression is a well-known integral representation of the modified Bessel function $K_{1/3}(v)$.[12] Using this expression, we obtain

$$\widetilde{I}_1 \approx \frac{2^{3/2}}{(3m)^{1/2}} K_{1/3}\left(\frac{2^{3/2}}{3} \frac{|\Delta|}{m^{1/2}}\right). \tag{A3}$$

In conclusion, we note that the accuracy of the expression (A3) is quite high, since the higher-order terms in the expansion of the function $F(\xi|m)$ in powers of ξ (terms $\sim \xi^5$) contribute only small corrections (of the order of $|\Delta|^{-2/3}$) to the result (A3).

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