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FERMIONIC MICROSTATES WITHIN THE PAINLEVÉ–GULLSTRAND BLACK HOLE

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We consider the quantum vacuum of a fermionic field in the presence of a black-hole background as a possible candidate for the stabilized black hole. The stable vacuum state (as well as thermal equilibrium states at an arbitrary temperature) can exist if we use the Painlevé–Gullstrand description of the black hole and the superluminal dispersion of the particle spectrum at high energy, which is introduced in the free-falling frame. This choice is inspired by the analogy between the quantum vacuum and the ground state of quantum liquid, in which the event horizon for the low-energy fermionic quasiparticles can also arise. The quantum vacuum is characterized by the Fermi surface that appears behind the event horizon. We do not consider the back reaction, and therefore, there is no guarantee that the stable black hole exists. But if it does exist, the Fermi surface behind the horizon would be the necessary attribute of its vacuum state. We also consider the exact discrete spectrum of fermions inside the horizon, which allows us to discuss the problem of fermion zero modes.

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1. INTRODUCTION

In 1981, Unruh proposed to study the black hole physics using its sonic analogue [1]. Originally suggested for classical liquids, this was later extended to quantum systems such as superfluids and Bose condensates [2–4]. The main advantage of the quantum liquids and gases is that in many respects, they are similar to the quantum vacuum of fermionic and bosonic fields. This analogy forms a view on the quantum vacuum as a special type of condensed matter — the «ether» — where the physical laws that we have at present can arise emergently as the energy or temperature of the «ether» decreases [5]. A particular scenario of the emergent formation of the effective gravity together with gauge fields and chiral fermions can be found in the recent review paper [6].

According to the topology in the momentum space, there are three types (universality classes) of the fermionic vacua. One of them has the trivial topology and its fermionic excitations are therefore fully gapped (massive fermions). The other two have a nontrivial momentum-space topology characterized by certain topological invariants in the momentum space [6]. One of the two nontrivial universality classes contains systems with Fermi points; their excitations are chiral fermions, whose energy vanishes at points in the momentum space. Another class represents systems with a wider manifold of zeroes: their gapless fermionic excitations are concentrated in the vicinity of the $2D$ surface in momentum space, the Fermi surface. This class contains Fermi liquids.

Here, we discuss the properties of the quantum vacuum in the presence of the event horizon. We assume that in the absence of the horizon, the fermionic vacuum belongs either to the trivial class (such as the Standard Model below the electroweak transition, where all fermions are massive) or to the class of Fermi points (such as the Standard Model above the electroweak transition, with its excitations being chiral massless fermions).

In the presence of a horizon, the region behind the horizon becomes the ergoregion: particles acquire negative energy there. In the true vacuum state, these

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negative-energy levels must be occupied, which means that the old vacuum must be reconstruced by filling these levels. We do not study the process of filling, which can be the smooth Hawking radiation process [7] or some other more violent process; we discuss the structure of the true vacuum state assuming that this state can be reached without destroying the horizon. In other words, we assume that the stable black hole can exist as a final ground state of the gravitational collapse. We find that behind the horizon, the fermionic vacuum belongs to the class of the Fermi surface.

The main sources for the appearance of the Fermi surface originate in the following properties of the event horizon. First, the emergence of Planck physics in the vicinity of (and behind) the horizon. The event horizon serves as a magnifying glass through which the phenomena at the Planck length scale could be visualized. At some scales, the Lorentz invariance — a property of the low-energy physics — inevitably becomes invalid and deviations from the linear (relativistic) spectrum become important. This violation of the Lorentz invariance is now popular in the literature [1, 9–13]. It leads to either subluminal or superluminal propagation at high energy, e.g.,

$$E^2(p) = c^2p^2(1 \pm p^2/p_p^2),$$

where p_p is the Planck momentum. In accordance with the condensed matter analogy, we assume that the high-energy (quasi)particles are superluminal, i.e., the sign is the plus. Because of the superluminal dispersion, there is a bottom in the Dirac sea, and the process of filling the negative-energy levels is therefore limited. When all of these levels are occupied, we come to a global vacuum state (or the global thermodynamical equilibrium with a positive heat capacity, if the temperature is finite). Thus, the superluminal dispersion of the particle energy gives rise to the energetic stability of the vacuum in the presence of a black hole.

The second important consequence of the event horizon, due to which the vacuum belongs to the class of systems with the Fermi surface, is that the horizon violates the time reversal symmetry of the system: the incoming and outgoing particles have different trajectories. In condensed matter, the appearance of the Fermi surface due to the violation of the time reversal symmetry is a typical phenomenon (see, e.g., [8] and also Sec. 12.4 in Ref. [6]).

In Refs. [4, 14], a stable black hole is also considered that exhibits a finite positive heat capacity, an arbitrary temperature, and no Hawking radiation. But it is assumed there that the time reversal symmetry is not broken in the final state (or is actually restored in the final state). The existence of such a stable black hole with the unbroken time reversal symmetry is also supported by the condensed matter analogies [4, 15, 16], in which stable infinite-redshift surfaces arise. An example of the infinite-redshift surface with no time reversal symmetry breaking is also provided by the extremal black hole, whose condensed matter analogue is discussed in Sec. 12.6 of review [6]. In all these examples, the Fermi surface does not appear. The black hole ground states with the time reversal symmetry are in some sense exceptional (in the same manner as the extremal black hole), and we do not discuss them here.

2. STATIONARY METRIC WITH THE EXPLICITLY VIOLATED TIME REVERSAL SYMMETRY

The vacuum can be well-defined only if the metric is stationary. In general relativity, the stationary metric for the black hole is provided in the Painlevé–Gullstrand spacetime [17]. The line element of the Painlevé–Gullstrand metric is

$$ds^2 = -c^2dt^2 + (d\mathbf{r} - \mathbf{v}dt)^2 = \\ = -(c^2 - v^2)dt^2 - 2\mathbf{v}d\mathbf{r}dt + d\mathbf{r}^2 \ , \quad (1)$$

where

$$\mathbf{v}(\mathbf{r}) = \pm\hat{\mathbf{r}}c\sqrt{\frac{r_h}{r}}, \quad r_h = \frac{2MG}{c^2}. \quad (2)$$

Here, M is the mass of the hole, r_h is the radius of the horizon, and G is the Newton gravitational constant; the minus sign in Eq. (2) gives the metric for the black hole, while the plus sign characterizes the white hole. The time reversal operation $t \to -t$ transforms the black hole into the white whole. The stationary property of this metric and the fact that it describes the spacetime in both the exterior and interior regions, are very attractive features that were explored starting from Ref. [18] (see [19–21]; an extension of the Painlevé–Gullstrand spacetime to the rotating black hole can be found in Ref. [22]).

In the case of the black hole, the field $\mathbf{v}(\mathbf{r})$ has a simple interpretation: it is the velocity of the observer who freely falls along the radius towards the center of the black hole with zero initial velocity at infinity. The motion of the observer obeys the Newtonian laws all the way through the horizon,

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}, \quad (3)$$

and his velocity is therefore given by

$$\mathbf{v}(\mathbf{r}) \equiv \frac{d\mathbf{r}}{dt} = -\hat{\mathbf{r}}\sqrt{\frac{2GM}{r}} \ . \tag{4}$$

The time coordinate t is the local proper time for the observer who drags the inertial coordinate frame with him.

As was first noticed by Unruh [1], the effective metric of type (1) is experienced by quasiparticles propagating in moving fluids. The field $\mathbf{v}(\mathbf{r})$ is then the velocity field of the liquid and c is the «maximum attainable velocity» of quasiparticles in the low-energy limit, for example the speed of sound in the case of phonons (see also [23–26, 6]). The horizon could be produced in liquids when the flow velocity becomes greater than c. The black hole and the white hole can be reproduced by the liquid flowing radially inward and outward, respectively. This is an explicit realization of the time reversal symmetry breaking by a flowing liquid: the time reversal operation reverses the direction of the flow of the «vacuum»,

$$\mathcal{T}\mathbf{v}(\mathbf{r}) = -\mathbf{v}(\mathbf{r}).$$

This Painlevé–Gullstrand spacetime, although not static, is stationary. That is why the energy $\tilde{E}$ of a (quasi)particle in this spacetime is determined in both the exterior and the interior regions. It can be obtained as the solution of the equation

$$g^{\mu\nu}p_\mu p_\nu + m^2 = 0$$

with $p_0 = -\tilde{E}$, which gives

$$\tilde{E}(\mathbf{p}) = E(p) + \mathbf{p}\cdot\mathbf{v}(\mathbf{r}) \ , \tag{5}$$

where $E(p)$ is the energy of the particle in the free-falling frame,

$$E^2(p) = p^2c^2 + m^2 \ . \tag{6}$$

For the «sonic» black hole, it is the energy of the quasiparticle in the frame comoving with the superfluid vacuum.

We now consider a massless (quasi)particle moving in the radial direction from the black hole horizon to infinity, i.e., with a positive radial momentum p_r. Because the metric is stationary, the energy of a particle in the Painlevé–Gullstrand frame (or of a quasiparticle in the laboratory frame) is conserved and we have $\tilde{E}$ = const. Its energy in the free-falling (superfluid comoving) frame is then given by

$$E(p) = cp_r = \frac{\tilde{E}}{1 + v(r)/c} = \frac{\tilde{E}}{1 - \sqrt{r_h/r}}. \tag{7}$$

This energy, which is very big near the horizon, decreases as the (quasi)particle moves away from the horizon. This is the gravitational red shift superimposed on the Doppler effect [27], because the emitter is freely falling with the velocity $v = v_s(r)$. The frequency of the spectral line measured by the observer at infinity is

$$\tilde{\omega} = \omega\sqrt{-g_{00}}\ \frac{\sqrt{1 - v^2/c^2}}{1 - v/c} = \omega\left(1 - \sqrt{\frac{r_h}{r}}\right) \ , \tag{8}$$

where ω is the nominal frequency of this line. The surface $r = r_h$ is the infinite redshift surface, and the energy in Eq. (7) diverges there. This means that if we observe particles coming to us from a very close vicinity of the horizon, these outgoing particles originally had a huge energy approaching the Planck energy scale. The event horizon can therefore serve as a magnifying glass that allows us to see what happens at the Planck length scale. At some point, the low-energy relativistic approximation inevitably becomes invalid and the Lorentz invariance is violated.

In quantum liquids, the nonlinear dispersion enters the velocity-independent energy $E(p)$ in the superfluid comoving frame. Taking the analogy with quantum liquids into account, we assume that in our vacuum, the Planck physics also enters the energy in the free-falling frame. The energy spectrum of particles is therefore given by Eq. (5), where

$$E^2(p) = m^2 + p^2c^2\left(1 \pm \frac{p^2}{p_p^2}\right). \tag{9}$$

As for the incoming massless particle, its radial momentum $p_r < 0$, and its energy in the comoving frame is therefore given by

$$E(p) = -cp_r = \frac{\tilde{E}}{1 - v(r)/c} = \frac{\tilde{E}}{1 + \sqrt{r_h/r}}. \tag{10}$$

It has no pathology at the horizon: the observer falling freely across the horizon sees no inconveniences when he crosses the horizon, and the Planck physics is therefore not evoked here.

The pathology reappears when one tries to construct the thermal global equilibrium state (or the vacuum state) in the presence of a horizon. In the global equilibrium, according to the Tolman law, the temperature measured by an observer in the comoving frame diverges at the horizon,

$$T(\mathbf{r}) = \frac{T_{Tolman}}{\sqrt{-g_{00}(\mathbf{r})}} = \frac{T_{Tolman}}{\sqrt{1 - v^2/c^2}}. \tag{11}$$

At some point, this temperature again becomes so high that the Planck physics becomes relevant. In the presence of a horizon, the global equilibrium is possible only for the superluminal dispersion, i.e., for the plus sign in Eq. (9). The reason is as follows. Behind the horizon, at $r < r_h$, the frame-dragging velocity exceeds the speed of light. In the relativistic domain, this implies that the radial coordinate r becomes time-like, because a (quasi)particle can move along the r coordinate in only one direction behind the horizon, towards the singularity. However, with the plus sign for the energy spectrum in Eq. (9), the (quasi)particles can go back and forth even behind the horizon. The space-like nature of the r coordinate is therefore restored by the superluminal dispersion and the global equilibrium becomes possible.

Finally, the condensed matter analogue of the formation of quantum field theory as an emergent phenomenon at low energy suggests that our vacuum is fermionic, while all the bosonic degrees of freedom can be obtained as collective modes of the fermionic vacuum. It is the Pauli principle for fermions that allows us to construct a stable vacuum in the presence of a horizon. Thus, there are three main necessary conditions for the existence of a stable vacuum with the broken time reversal symmetry in the presence of a black hole: the vacuum is fermionic, its fermionic excitations have superluminal dispersion, and the black hole is described by the Painlevé–Gullstrand metric. All the three conditions are motivated by the quantum liquid similarities.

3. THE DIRAC EQUATION IN THE PAINLEVÉ–GULLSTRAND METRIC

In Ref. [28], fermions were considered in the semiclassical approximation. Here, we extend this analysis to the exact quantum-mechanical one. In the presence of a nontrivial gravitational background, fermions are described by the tetrad formalism. We here follow Ref. [29]. The metric $g_{\mu\nu}$ can be written in terms of the tetrad e^a_μ as

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}, \tag{12}$$

where $\eta^{ab} = \mathrm{diag}(-1, 1, 1, 1)$. The Dirac equation in a curved spacetime is

$$(i\gamma^a E^\mu_a D_\mu - m)\,\Psi = 0, \quad D_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu;ab}\gamma^a\gamma^b, \tag{13}$$

where the dual tetrad field E^ν_b obeys

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}, \quad E^\mu_a e^a_\alpha = \delta^\mu_\nu, \quad E^\mu_a E^\nu_b \eta^{ab} = g^{\mu\nu}, \tag{14}$$

$$e^a_\mu = g_{\mu\nu}\eta^{ab}E^\nu_b, \quad e_{\nu b} = e^a_\nu \eta_{ab} = g_{\mu\nu}E^\mu_b, \tag{15}$$

and the torsion field is

$$\omega_{\mu;ab} = E^\nu_a \eta_{bc}\nabla_\mu e^c_\nu = E^\nu_a \nabla_\mu\left(g_{\nu\alpha}E^\alpha_b\right) = \\ = E^\nu_a \nabla_\mu e_{\nu b} = E^\nu_a\left(\partial_\mu e_{\nu b} - \Gamma^\gamma_{\mu\nu}e_{\gamma b}\right). \tag{16}$$

The vielbeins corresponding to the general «flow» metric in Eq. (1) are

$$e^a_\mu = \delta^a_\mu + \tilde{e}^a_\mu, \quad \tilde{e}^a_\mu = v^i\delta^a_i\delta^0_\mu. \tag{17}$$

The only nonzero correction to the tetrad field δ^a_μ for Minkowski spacetime is

$$\tilde{e}^i_0 = v^i \neq 0.$$

For the Painlevé–Gullstrand metric of the black hole in spherical coordinates, we have

$$\begin{aligned} e^0_\mu &= (1, 0, 0, 0), \quad e^1_\mu = (v, 1, 0, 0), \\ e^2_\mu &= (0, 0, r, 0), \quad e^3_\mu = (0, 0, 0, r\sin\theta), \end{aligned} \tag{18}$$

where $v(r) = -r^{-1/2}$, assuming that $c = r_h = 1$.

The violation of the Lorentz invariance at high energy can be introduced by adding a nonlinear γ_5-term that leads to the superluminal dipersion. As a result, we obtain the Dirac equation in the Painlevé–Gullstrand metric [22], which is now modified by a non-Lorentzian term,

$$i\partial_t\Psi = -ic\alpha^i\partial_i\Psi + m\gamma_0\Psi + H_p\Psi + H_g\Psi. \tag{19}$$

Here, H_p and H_g are the respective Hamiltoinians coming from the Planck physics and from the gravitational field,

$$H_p = -\frac{c}{p_p}\gamma_5\partial^2_i, \quad H_g = ic\sqrt{\frac{r_h}{r}}\left(\frac{3}{4r} + \partial_r\right). \tag{20}$$

The γ matrices that we use are given by

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{21}$$

and

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{22}$$

After the multiplication by $r_h/\hbar c$, we obtain a dimensionless form and write $\hbar = c = r_h = 1$ and $p_0 = p_p r_h/\hbar \gg 1$.

4. FERMIONIC EIGENSTATES IN THE PAINLEVÉ–GULLSTRAND BLACK HOLE

Because ∂_t is a timelike Killing vector in the Painlevé–Gullstrand black hole, the energy $\tilde{E}$ is a well-defined quantity and the variables t and $\mathbf{r}$ can be separated by writing

$$\Psi = \begin{pmatrix} \phi(\mathbf{r}) \\ \chi(\mathbf{r}) \end{pmatrix} e^{-i\tilde{E}t}. \tag{23}$$

The $\mathbf{r}$-equations are now given by

$$\begin{aligned} \tilde{E}\phi &= \sigma\cdot\mathbf{p}\chi + m\phi - i\frac{1}{p_0}p^2\chi + H_g\phi, \\ \tilde{E}\chi &= \sigma\cdot\mathbf{p}\phi - m\chi + i\frac{1}{p_0}p^2\phi + H_g\chi, \end{aligned} \tag{24}$$

where $p_i = -i\partial_i$. Using the spherical symmetry, we introduce spherical harmonics in the standard way. These are eigenstates of the operators $\mathbf{J}^2$ and J_z, where $\mathbf{J}$ is the total angular momentum,

$$J_i = L_i + S_i = L_i + \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \tag{25}$$

and L_i is the orbital angular momentum operator in R^3. Because we are interested in the states with high momenta $J \sim p_0 \gg mr_h/\hbar$, we can neglect the mass term. We then obtain the ansatz

$$\phi_{J,J_3} = \frac{1}{2r} \times \\ \times \left((f^+(r)+f^-(r))\Omega_l + (f^+(r)-f^-(r))\Omega_{l+1}\right), \tag{26}$$

$$\chi_{J,J_3} = \\ = \frac{1}{2r}\left((g^+(r)-f^+(r))\Omega_l + (g^+(r)+g^-(r))\Omega_{l+1}\right), \tag{27}$$

where the spherical harmonics are given by

$$\Omega_l = \begin{pmatrix} \sqrt{\dfrac{J+J_z}{2J}}Y_{l,J_z-1/2} \\ \sqrt{\dfrac{J-J_3}{2J}}Y_{l,J_3+1/2} \end{pmatrix}, \\ \Omega_{l+1} = \begin{pmatrix} -\sqrt{\dfrac{J-J_3+1}{2J+2}}Y_{l+1,J_3-1/2} \\ \sqrt{\dfrac{J+J_3+1}{2J+2}}Y_{l+1,J_3+1/2} \end{pmatrix}, \tag{28}$$

with $l = J - 1/2$. The radial functions satisfy the equations

$$\tilde{E}\begin{pmatrix} f^+ \\ g^+ \end{pmatrix} = \Bigg[i\partial_r \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \\ + i\frac{l+1}{r}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{l+1}{p_0 r^2} + \\ + \frac{1}{p_0}\left(-\partial_r^2 + \frac{(l+1)^2}{r^2}\right)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \\ + i\sqrt{1/r}\left(\partial_r - \frac{1}{4r}\right)\Bigg]\begin{pmatrix} f^+ \\ g^+ \end{pmatrix}, \tag{29}$$

$$\tilde{E}\begin{pmatrix} f^- \\ g^- \end{pmatrix} = \Bigg[i\partial_r \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \\ + i\frac{l+1}{r}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{l+1}{p_0 r^2} + \\ + \frac{1}{p_0}\left(-\partial_r^2 + \frac{(l+1)^2}{r^2}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \\ + i\sqrt{1/r}\left(\partial_r - \frac{1}{4r}\right)\Bigg]\begin{pmatrix} f^- \\ g^- \end{pmatrix}. \tag{30}$$

Taking the complex conjugation of (29), we obtain Eq. (30) with the reversed sign of energy. This implies that the matrices cannot be diagonalized simultaneously unless $\tilde{E} = 0$, and therefore, either (f^+, g^+) or (f^-, g^-) is nonzero for the eigenstate with $\tilde{E} \neq 0$.

Equations (29) and (30) are the starting point for our analysis of the fermionic vacuum and excitations.

5. FERMIONS IN THE SEMICLASSICAL APPROXIMATION

In the classical limit, with $(f,g) \propto \exp\left(i\int p_r dr\right)$, we obtain the energy spectrum

$$\left(\tilde{E} + \frac{p_r}{\sqrt{r}}\right)^2 = p_r^2 + \frac{l^2}{r^2} + \frac{1}{p_0^2}\left(p_r^2 + \frac{l^2}{r^2}\right)^2, \tag{31}$$

where we neglected small terms of the relative order $1/p_0$. We are interested in the states with the lowest energy, because they give the main contribution to thermodynamics. For a given l, the energy of the fermion becomes zero at the following values of the radial momentum:

$$p_r^2(r, \tilde{E}=0, l) = \frac{1}{2r}p_0^2(1-r) - \frac{l^2}{r^2} \pm \\ \pm \frac{1}{r}\sqrt{\frac{1}{4}p_0^4(1-r)^2 - \frac{p_0^2 l^2}{r}}. \tag{32}$$

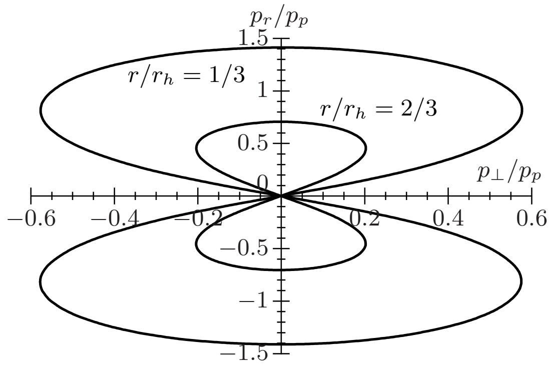

Fig. 1. Fermi surface $\tilde{E}(\mathbf{p}) = 0$ at two positions inside the black hole: $r = 2r_h/3$ and $r = r_h/3$

This coincides with Eq. (13) in [28], where the semiclassical approximation was used from the very beginning.

Within the completely classical analysis, with $p_\perp = l/r$ representing the transverse momentum of the fermion, Eq. (31) at $\tilde{E} = 0$ gives the closed $2D$ surface in the $3D$ momentum space. This surface, on which the energy of particles is zero, represents the Fermi surface; it exists only inside the horizon, i.e., at $r < r_h$ $(r < 1)$. Figure 1 demonstrates the Fermi surface $\tilde{E}(\mathbf{p}) = 0$ at two values of the radius r behind the horizon: $r = 2r_h/3$ and $r = r_h/3$. The area of the Fermi surface increases with decreasing r.

In the true ground state, all the levels inside the Fermi surface (i.e., those with $\tilde{E}(\mathbf{p}) < 0$) must be occupied. Of course, this reconstruction of the vacuum involving the Planck energy scale can have tremendous consequences for the black hole itself. These cannot be described by the phenomenological low-energy physics. Nevertheless, we can claim that if the horizon survives the vacuum reconstruction, the Fermi surface also survives because of its topological robustness. In this case, the statistical physics of the black hole microstates is entirely determined by the fermionic states in the vicinity of the Fermi surface. In particular, the entropy and the heat capacity of the black hole are linear in the temperature T,

$$S = C = \frac{\pi^2}{3} N(0) T, \tag{33}$$

where $N(0)$ is the density of states at $\tilde{E} = 0$. From the general dimensionality arguments together with the fact that the density of states must be proportional to the volume of the Fermi liquid, we obtain

$$N(0) = \gamma N_F \frac{p_p^2 r_h^3}{\hbar^3 c}\ , \tag{34}$$

where N_F is the number of fermionic species and γ is a dimensionless constant of order of unity. In our oversimplified model, $\gamma = 4/35\pi$ [28].

In the interior region, the equation of state is

$$p = \rho \propto T^2.$$

Incidentally, this coincides with the equation of state of the perfect fluid inside the horizon required to obtain the Bekenstein–Hawking entropy (see Refs. [30, 31] and [14]). In the Sakharov induced gravity [32], the Planck momentum and the gravitational constant are related by $N_F p_p^2 \sim \hbar c^3/G$. This actually implies that the microscopic parameters of the system, the fermion number N_F and the Planck momentum p_p, are combined to form the phenomenological parameter of the effective theory, the gravitational constant G. If we assume that only the thermal fermions are gravitating, we obtain

$$M \sim \int dV \rho \sim T^2 M^3 G^2.$$

This gives estimates for the temperature and entropy of the black hole,

$$T \sim 1/GM, \quad S \sim GM^2,$$

which are in correspondence with the Hawking–Bekenstein entropy and the Hawking temperature. Only the phenomenological parameters G and c are involved here, while the microscopic parameters N_F and p_p drop out. This is in agreement with the observation made by Jacobson [33] that the black hole entropy and the gravitational constant are renormalized such that the relation between them is preserved. All this means that statistical properties of the black hole can be produced by the Fermi liquid in the interior of the black hole.

6. EXACT ENERGY LEVELS

Another problem that can be investigated using our scheme is that of the fermion is zero modes: are there fermionic modes that have exactly zero energy in the exact quantum mechanical problem? If yes, this would justify the conjectures that the black hole has a nonzero entropy even at $T = 0$, and also that the area of the black hole is a quantized quantity [34–36]. For this reason, we now proceed to solving eigenvalue equations (29) and (30).

It is impossible to solve these equations analytically, but one can choose the region of parameters where they can be solved using the perturbation theory expansion in the small parameter $1/p_0$. To find this region, we

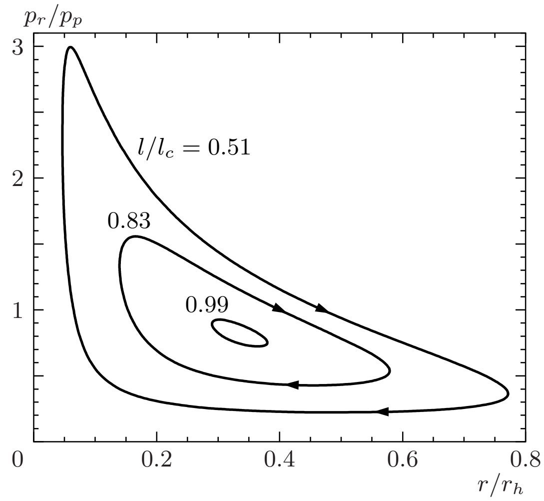

Fig. 2. Closed trajectories of the radial motion inside the black hole at zero energy $\tilde{E} = 0$ for different values of the angular momentum l

consider semiclassical trajectories of the radial motion $p_r(r)$ at $\tilde{E} = 0$ for different l, Eq. (32). These trajectories are shown in Fig. 2 (we used $p_0 = 10000$). If l is small compared to p_0, these trajectories are highly asymmetric: the incoming and outgoing particles experience essentially different motions. The conventional relativistic particles with a small momentum compared to the Planck momentum p_p can move only towards the singularity. However, when they acquire a large momentum, the nonlinear dispersion allows them to move away from the singularity. As a result, the trajectories of particles become closed. This asymmetry reflects the violation of the time reversal symmetry by the horizon.

However, as l increases, the trajectories become more and more symmetric. Near the maximum value

$$l^{(c)} = 3^{-3/2} p_0, \tag{35}$$

they become perfectly elliptic and increasingly more concentrated in the vicinity of the center point

$$r^{(c)} = \frac{1}{3}, \tag{36}$$

$$p^{(c)} = \pm\sqrt{\frac{2}{3}}p_0. \tag{37}$$

This implies that in vicinity of $r^{(c)}$ and $p^{(c)}$, the Hamiltonian describing the radial motion becomes that of oscillators. We can therefore expand the equations in the vicinity of $p^{(c)}$ and $r^{(c)}$ using the small parameter $1/p_0$,

$$\begin{aligned} r &= r^{(c)} + x, \\ p_r &= p^{(c)} - i\partial_x \ . \end{aligned} \tag{38}$$

It can be seen that the regions where x and ∂_x are concentrated

$$x \propto \frac{1}{\sqrt{p_0}} \ll r^{(c)}, \quad \partial_x \propto \sqrt{p_0} \ll |p^{(c)}|, \tag{39}$$

become really small compared to $r^{(c)}$ and $p^{(c)}$ as p_0 increases. As a result, after lengthy but straightforward expansion of Eq. (29) near the point with $p^{(c)} > 0$, we obtain (keeping the terms of the order of unity) the effective oscillator Hamiltonian

$$\begin{aligned} H_{eff} = -3\sqrt{\frac{3}{2}}\delta l + \frac{13p_0}{2\sqrt{2}}x^2 + \frac{2\sqrt{2}}{3p_0}p^2 + \\ + \frac{5}{2\sqrt{3}}(xp + px) + \frac{3\sqrt{3}}{4}, \end{aligned} \tag{40}$$

where

$$\delta l \equiv l^{(c)} - (l+1). \tag{41}$$

Diagonalization gives the energy spectrum

$$\tilde{E}_1 = -3\sqrt{\frac{3}{2}}\delta l + 3n_r + \frac{3}{2} + \frac{3\sqrt{3}}{4}, \tag{42}$$

where $n_r = 0, 1, \ldots$ is the radial quantum number. Accordingly, the expansion near the point with $p^{(c)} < 0$ and the same procedure for Eq. (30) give the other three sets of the energy levels,

$$\tilde{E}_2 = 3\sqrt{\frac{3}{2}}\delta l - 3n_r - \frac{3}{2} + \frac{3\sqrt{3}}{4}\ , \tag{43}$$

$$\tilde{E}_3 = -3\sqrt{\frac{3}{2}}\delta l + 3n_r + \frac{3}{2} - \frac{3\sqrt{3}}{4} = -\tilde{E}_2 \tag{44}$$

and

$$\tilde{E}_4 = 3\sqrt{\frac{3}{2}}\delta l - 3n_r - \frac{3}{2} - \frac{3\sqrt{3}}{4} = -\tilde{E}_1. \tag{45}$$

Finally, in dimensionful units, we have the discrete levels of fermions in the vicinity of the Fermi surface,

$$\begin{aligned} \tilde{E}(J, n_r) = \pm\frac{\hbar c}{r_h} \times \\ \times \left(\frac{1}{\sqrt{2}}\frac{p_p r_h}{\hbar} - 3\sqrt{\frac{3}{2}}\left(J + \frac{1}{2}\right) - 3n_r - \frac{3}{2} \pm \frac{3\sqrt{3}}{4}\right), \end{aligned} \tag{46}$$

where all the four signs must be taken into account. This equation is valid for J smaller than but close to the maximum value

$$J^{(c)} = p_p r_h / 3\sqrt{3}\hbar$$

at which zero-energy states can still exist.

Equation (46) allows us to conclude that the true fermion zero modes exist in the presence of a black hole. For general values of $p_p r_h$, and hence, for the general values of the black hole area $A = 4\pi r_h^2$, there are no states with exactly zero energy. A zero-energy eigenstate can be found for some special values of A. However, because of the incommensurability between the radial and orbital quantum numbers, the degeneracy of the $\tilde{E} = 0$ levels is small, and the fermion zero modes cannot therefore produce the entropy at $T = 0$ that is proportional to the area of the horizon. Accordingly, there are no microscopic reasons for the quantization of the area of the horizon.

There are no topological arguments ensuring the existence of the exact fermion zero modes. On the other hand, the momentum-space topology prescribes the existence of zero-energy fermion modes at the semiclasical level. These modes form a surface in the momentum space — the Fermi surface — in Fig. 1. The existence of the Fermi surface is a robust property of the fermionic vacuum; the Fermi surface survives when the back reaction is introduced (of course, if the horizon survives). It is the Fermi liquid whose thermal states give rise to the entropy proportional to the area, as was discussed in the previous section.

7. CONCLUSIONS

In deriving the fermionic microstates responsible for the statistical mechanics of the black hole, we used an analogy between quantum liquids and the quantum vacuum, the ether. We know that there are two preferred reference frames in superfluids. One of them is the «absolute» spacetime (x, t) of the laboratory frame, which can be Galilean as well as Minkowskian with c being the real speed of light. In the effective gravity experienced by the low-energy excitations in quantum liquids, the effective «acoustic» metric $g_{\mu\nu}^{acoustc}$ appears as a function of this «absolute» spacetime (x, t). The other preferred reference frame is the local frame, where the metric is Minkowskian in the acoustic sense, i.e., with c being the maximum attainable speed of low-energy quasiparticles. This frame is comoving with the superfluid condensate. In this frame, the energy spectrum does not depend on the velocity $\mathbf{v}$ of the condensate and has the form given in Eq. (9). It is therefore in this frame that the Planck energy physics is properly introduced: if the energy becomes big in the superfluid comoving frame, the acoustic Lorentz symmetry is violated.

As for the quantum vacuum, the attainable energies are still so low that we cannot select the preferred reference frame. In particular, we cannot say in which reference frame the Planck energy physics must be introduced, and whether there is an absolute spacetime. The magnifying glass of the event horizon can serve as a possible source of spotting these reference frames.

In our low-energy corner, the Einstein action is covariant: it does not depend on the choice of the reference frame. That is why the Einstein equations can be solved in any coordinate system. However, in the presence of a horizon or ergoregion, some of the solutions are not defined in the entire spacetime of the quantum vacuum. In these cases, the discrimination between different solutions arises and one must choose between them. In quantum liquids, the choice is natural because the absolute coordinates are used from the very beginning. But in general relativity, the ambiguity in the presence of a horizon imposes the problem of properly choosing the solution. This problem cannot be solved within the effective theory, while the fundamental «microscopic» background is still not known, and one can only guess the proper solution of Einstein equations using which the vacuum state can be constructed.

It is clear that the Schwarzschild solution is not the proper choice, in particular because the entire spacetime is not covered by the Schwarzschild coordinates. According to the quantum liquid analogy, the Painlevé–Gullstrand metric with the inward frame dragging can be a reasonable choice. Its analogue can be really reproduced (at least in principle) in quantum liquids. The analogy also suggests that the Painlevé–Gullstrand spacetime can be considered as the absolute one in which the true vacuum must be determined. On the other hand, the local frame of the free-falling observer can be considered as an analogue of the superfluid comoving frame in which the Planck energy physics must be introduced. We again warn that this choice cannot be justified from the standpoint of the effective theory alone.

If the Planck physics is in addition superluminal, as is also suggested by the quantum liquid analogy, the stable quantum vacuum can even be constructed in the presence of a horizon. We argue that the main property of such a quantum vacuum, distinguishing it from the original vacuum of the Standard Model, is the existence of the Fermi surface inside the horizon. The statistical mechanics of the Fermi liquid formed inside the horizon is responsible for the thermodynamics of the black hole.

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REFERENCES

1. W. G. Unruh, Phys. Rev. Lett. **46**, 1351 (1981); Phys. Rev. D **51**, 2827 (1995).

2. T. Jacobson and G. E. Volovik, Phys. Rev. D **58**, 064021 (1998).

3. L. J. Garay, J. R. Anglin, J. I. Cirac and P. Zoller, Phys. Rev. Lett. **85**, 4643 (2000); Phys. Rev. A **63**, 023611 (2001).

4. G. Chapline, E. Hohlfeld, R. B. Laughlin, and D. I. Santiago, Phil. Mag. B **81**, 235 (2001).

5. R. Laughlin and D. Pines, Proc. Natl. Acad. Sc. USA **97**, 28 (2000).

6. G. E. Volovik, Phys. Rep. **351**, 195 (2001).

7. S. W. Hawking, Nature **248**, 30 (1974).

8. G. E. Volovik, Phys. Lett. A **142**, 282 (1989).

9. S. Corley and T. Jacobson, Phys. Rev. D **54**, 1568 (1996).

10. S. Corley, Phys. Rev. D **57**, 6280 (1998).

11. S. Corley and T. Jacobson, Phys. Rev. D **59**, 124011 (1999).

12. A. Starobinsky, Pis'ma v Zh. Eksp. Teor. Fiz. **73**, 415 (2001).

13. T. Jacobson, E-print archives gr-qc/0110079; T. Jacobson and D. Mattingly, Phys. Rev. D **63**, 041502 (2001); E-print archives gr-qc/0007031.

14. P. O. Mazur and E. Mottola, E-print archives gr-qc/0109035.

15. M. Mohazzab, J. Low Temp. Phys. **121**, 659 (2000).

16. G. E. Volovik, Pis'ma v Zh. Eksp. Teor. Fiz. **70**, 711 (1999).

17. P. Painlevé, C. R. Hebd Acad. Sci. (Paris) **173**, 677 (1921); A. Gullstrand, Arkiv. Mat. Astron. Fys. **16**, 1 (1922).

18. P. Kraus and F. Wilczek, Mod. Phys. Lett. A **9**, 3713 (1994).

19. K. Martel and E. Poisson, Am. J. Phys. **69**, 476 (2001).

20. R. Schützhold, Phys. Rev. D **64**, 024029 (2001).

21. M. K. Parikh and F. Wilczek, Phys. Rev. Lett. **85**, 5042 (2000).

22. C. Doran, Phys. Rev. D **61**, 067503 (2000).

23. M. Visser, Class. Quantum Grav. **15**, 1767 (1998).

24. S. Liberati, S. Sonego, and M. Visser, Class. Quant. Grav. **17**, 2903 (2000).

25. M. Stone, E-print archives cond-mat/0012316.

26. M. Sakagami and A. Ohashi, E-print archives gr-qc/0108072.

27. L. D. Landau and E. M. Lifshitz, *Classical Fields*, Pergamon Press, Oxford (1975).

28. G. E. Volovik, Pis'ma v Zh. Eksp. Teor. Fiz. **73**, 721 (2001).

29. S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley & Sons (1972.

30. W. H. Zurek and Don N. Page, Phys. Rev. D **29**, 628631 (1984).

31. G. 't Hooft, Nucl. Phys. Proc. Suppl. **68**, 174 (1998).

32. A. D. Sakharov, Dokl. Akad. Nauk **177**, 70 (1967).

33. T. Jacobson, E-print archives gr-qc/9404039.

34. J. D. Bekenstein, in: *Proc. of the 8^{th} Marcel Grossman Meeting*, ed. by Tsvi Piran, Singapore, World Scientific (1999); E-print archives gr-qc/9710076.

35. H. A. Kastrup, Phys. Lett. B **413**, 267 (1997).

36. V. F. Mukhanov, Pis'ma v Zh. Eksp. Teor. Fiz. **44**, 63 (1986).