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ANOMALOUS HYDRODYNAMICS OF FRACTIONAL QUANTUM HALL STATES

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Dedicated to the memory of Professor Anatoly Larkin

We propose a comprehensive framework for quantum hydrodynamics of the fractional quantum Hall (FQH) states. We suggest that the electronic fluid in the FQH regime can be phenomenologically described by the quantized hydrodynamics of vortices in an incompressible rotating liquid. We demonstrate that such hydrodynamics captures all major features of FQH states, including the subtle effect of the Lorentz shear stress. We present a consistent quantization of the hydrodynamics of an incompressible fluid, providing a powerful framework to study the FQH effect and superfluids. We obtain the quantum hydrodynamics of the vortex flow by quantizing the Kirchhoff equations for vortex dynamics.

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1. INTRODUCTION

Quantum systems with the effectively strong interaction form liquids whose flows are coherent quantum collective motions. Among them, there are interesting notable cases where such liquids allow a hydrodynamics description. That is when the long-wave, slow flows can be effectively described solely in terms of a macroscopic, but quantum, pair of canonical fields of density $\rho(r,t)$ and velocity $v(r,t)$. Such quantum flows are the subject of quantum hydrodynamics. In the classical case, the principle of local equilibrium reduces the Boltzmann kinetic equation for the distribution function to the hydrodynamics equations for the density and the velocity (see, e. g., [1]). Local equilibrium occurs when the characteristic time of the flow exceeds the characteristic time of collisions, and the characteristic scale of the flow exceeds the mean free path of particles. A quantum analog of the principle of local equilibrium is yet to be understood, but when it comes to effect, it involves long-range coherent effects. Strong coherence emerges as a result of interactions. Notable examples of quantum hydrodynamics are superfluid helium, superconductors, trapped cooled atomic gases, and Luttinger liquids. A fractional quantum Hall (FQH) liquid is yet another case.

Electronic states confined within the lowest Landau level by the quantizing magnetic field are holomorphic. The holomorphic nature of states makes the hydrodynamic description possible.

A quest for the hydrodynamics of a FQH liquid originated in a seminal paper [2]. Earlier approaches to FQH states in Refs. [3–5] are somewhat related to the hydrodynamics, as noted in Ref. [6]. Hydrodynamics of FQH states is in the focus of a renewed interest.

In hydrodynamics, a few basic principles, symmetries, and a few phenomenological parameters are sufficient to formulate the fundamental equations. In the case of the FQH effect (FQHE), we already possess sufficient characterizations of states. They can be used as a basis of the hydrodynamics approach. For this, a microscopical Hamiltonian and a deeper understanding of the underlying microscopic mechanisms of emergence of correlated liquid states are, in fact, not necessary.

In this paper, we formulate a minimal number of principles sufficient to develop the hydrodynamics of FQH bulk states in a close similarity to Feynman's theory of superfluid helium [7], and the magneto-roton theory of collective excitations in FQH states in Ref. [2]. We discuss only the simplest Laughlin states. Elsewhere, we hope to be able to address the hydrodynamics of other, richer FQH states, possessing additional

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symmetries, such as the 5/2 state. They can be studied within the framework developed here.

We argue that states of the FQH liquid can be treated as flows of quantized vortices in a quantum incompressible rotating inviscid liquid. On this basis, we obtain the major features of the FQHE including subtle effects such as the Lorentz shear stress[1)], missed by the previous approaches [3–6]. In particular, the Laughlin wave function

$$\psi_0(z_1,\ldots,z_N) = \exp\left(-\frac{1}{4\ell^2}\sum_i |z_i|^2\right) \times \prod_{i>j}(z_i - z_j)^\beta, \quad \beta = \frac{1}{\nu}, \quad (1)$$

emerges as the ground state of the vortex fluid. Here, $\ell = \sqrt{\hbar c/eB}$ is the magnetic length.

To the author's knowledge, a hydrodynamics of vortex flows has not been developed. It is an interesting subject in and of itself. Apart from the FQHE, it is also relevant in the theory of superfluids and classical hydrodynamics. In this paper, we present a consistent quantum hydrodynamics of such a fluid.

The hydrodynamics of vortex matter differs from the Euler hydrodynamics. Its quantum version differs from the canonical quantum hydrodynamics of Landau [13]. The major difference is the anomalous terms. These terms represent the Lorentz shear force. The emergence of such forces in hydrodynamics, classical and quantum alike, is the major focus of this paper.

In Sec. 3, we start from the observation that the FQH states can be interpreted as the states of quantized Kirchhoff vortex matter and then develop the hydrodynamics of vortex matter in Sec. 4. We summarize the main results in Sec. 5, and then give the details of derivations in Secs. [6–9].

Some results presented below were obtained in collaboration with Alexander Abanov. This paper is an extended version of Ref. [14].

2. FOUNDATIONAL PRINCIPLES OF HYDRODYNAMICS OF THE FQH LIQUID

2.1. Characterization of fractional quantum Hall states

Electrons in a quantizing magnetic field confined in $2D$ heterostructures in the regime dominated by the Coulomb interaction form FQH states. The most robust FQH states occur at the filling fraction $\nu = 1/3$; that are the Laughlin states. The FQH states form a quantum liquid. This liquid can be characterized as follows.

- Flows are incompressible [15], and almost dissipation-free [16, 17].
- The spectrum of bulk excitations is gapped [2, 17]. The gap is less than the cyclotron energy, $\hbar\omega_c > \Delta_\nu$. Only edge states — excitations localized on the boundary — are soft [18].
- The Hall conductance is fractionally quantized [16].
- Elementary excitations in the bulk of the fluid are vortices. Vortices carry fractionally quantized negative electronic charge [15].

More subtle features recently discussed in the literature are as follows.

- Edge excitations consist of two branches of nonlinear solitons: subsonic solitons with a fractional negative electronic charge and supersonic solitons with the unit electronic charge [19].
- Quantized double layers of the density at boundaries and vortices [19].
- The Lorentz shear stress and anomalous viscosity (or odd viscosity, or Hall viscosity) [8–12].

From the listed properties, we select a set of the foundational principles and attempt to obtain others as consequences. The set of basic principles is remarkably small. We only assume that electrons in the FQH regime form a quantum fluid and that the fluid is incompressible and flows possess a macroscopic number of equally oriented vortices.

We refer to such flow as chiral flow. Since in a quantum fluid, vorticity is quantized, a unit volume of the fluid contains the quantum of vorticity. We want to demonstrate that the chiral flow captures all known physics of the FQHE.

We start with a general discussion of scales of FQH (bulk) states.

2.2. Scales, holomorphic states, and incompressibility

There are two distinct energy scales: the cyclotron energy $\hbar\omega_c = e\hbar B/m_b c$, which defines the distance between Landau levels, and the gap in the bulk excitation spectrum Δ_ν. The former is determined by the band electronic mass m_b and by the magnetic field. The latter is a characteristic of the Coulomb energy. From the theoretical standpoint, the very existence of FQH states assumes that the cyclotron energy is larger than

[1)] For developments on this subject see [8–12].

the gap, $\Delta_\nu \ll \hbar\omega_c$. If this limit holds[2)], the flows with an energy E exceeding the gap can still be comprised of states on the lowest Landau level, $\Delta_\nu \ll E \ll \hbar\omega_c$. Such motion does not depend on the band electronic mass m_b.

We consider a small modulation of the electronic density $\rho(r)$ and ignore the electrostatic interaction of a nonuniform charged fluid[3)]. Such a flow has a momentum flux $P(r)$ and propagates with a velocity $v(r)$. We assume that at small modulations, the momentum flux is equal to $P = m_*\rho v$, where m_* is the inertia of the flow. It seems natural to assume that the inertia is set by the scale provided by the gap, $\Delta_\nu \sim \hbar^2/m_*\ell^2$. The mass m_* exceeds that of a band electron:

$$m_*/m_b \sim \hbar\omega_c/\Delta_\nu > 1.$$

Generally, waves propagating through the bulk of the FQH liquid are essentially nonlinear. However, stationary linear waves in the bulk are possible in the nonuniform electric and magnetic fields, or in a curved space. The sector of stationary linear waves some time is called topological.

Wave functions of states with the energy less than the the cyclotron energy (the lowest Landau level) are holomorphic. It is customary to describe the set of states on the lowest Landau level as the Bargmann space [20]. Coherent states of the Bargmann space are labeled by symmetric polynomials in the holomorphic coordinates of particles $z_i = x_i + iy_i$ and the holomorphic momenta

$$\partial_{z_i} = \frac{1}{2}\left(\partial_{x_i} - i\partial_{y_i}\right).$$

Let Q be such a polynomial and $Q^\dagger$ be the Hermitian conjugate polynomial, which depends on antiholomorphic coordinates $\bar z_i = x_i - iy_i$ and antiholomorphic momenta

$$\partial^\dagger_{z_i} = -\frac{1}{2}(\partial^T_{x_i} + i\partial^T_{y_i}).$$

The symbol "T" is the transposition. Then in the notation of the Bargmann space, the "bra" and "ket" states are

$$\langle Q| = \prod_{i>j}(\bar z_i - \bar z_j)^\beta Q^\dagger \exp\left(-\frac{1}{2\ell^2}\sum_i |z_i|^2\right), \qquad Q\prod_{i>j}(z_i - z_j)^\beta = |Q\rangle. \tag{2}$$

[2)] In experiments, the cyclotron energy is only a few times larger than the gap.

[3)] In FQH liquids, the Coulomb forces essentially block propagating waves in the bulk. In this paper, we neglect Coulomb forces in order to unmask laws of quantum hydrodynamics.

Flows within the first Landau level are incompressible. The term "incompressible flow" is sometimes attributed to the gapped spectrum. Rather, the incompressibility reflects the holomorphic nature of FQH states. This is seen from the following argument. For simplicity, we consider a coherent state characterized by a polynomial Q that depends only on coordinates z_i. The phase of the wave function of such a state differs from the phase of the ground state by the phase of the holomorphic polynomial $\operatorname{Im}\log Q$. Since the velocity is a gradient of the phase, the phase is a hydrodynamic potential. The phase is harmonic everywhere except points where the wave function vanishes. Since the wave function is single-valued, it vanishes as an integer power of holomorphic coordinates. Therefore, the allowed singularities of the phase correspond to quantized vortices. There are no sources, and hence the gradient of the phase is divergence-free,

$$\omega_c \to \infty: \quad \nabla\cdot v = 0. \tag{3}$$

There are two immediate consequences of incompressibility. One is that the material derivative of the density vanishes,

$$D_t\rho \equiv \left(\frac{\partial}{\partial t} + v\cdot\nabla\right)\rho = 0. \tag{4}$$

The other is that flows in homogeneous $2D$ incompressible liquids do not possess linear waves. Only available bulk flows are nonlinear flows of vorticity. The flow can be viewed as a motion of a neutral gas of quasiholes and quasiparticles.

In the next section, we identify the FQH states with vortices in a quantum incompressible rotating fluid.

3. KIRCHHOFF EQUATIONS

We start by recalling the classical Kirchhoff equations for rotating incompressible inviscid Euler flows with constant density (see e. g., [21]), and then proceed with the quantization.

3.1. Classical Kirchhoff equations for an incompressible fluid

In two dimensions, an incompressible fluid with a constant density is fully characterized by its vorticity. The curl of the Euler equation for the incompressible fluid with a constant density,

$$D_t u \equiv (\partial_t + u\cdot\nabla)u = -\nabla p, \tag{5}$$

yields a single (pseudo) scalar equation for the vorticity:

$$D_t(\nabla \times u) = 0. \tag{6}$$

In this form, the Euler equation has a simple geometrical meaning: the material derivative of the vorticity vanishes. Vorticity is transported along the divergence-free velocity field u.

Helmholtz, and later Kirchhoff realized that there is a class of solutions of vorticity equation (6) that consists of a finite number of point-like vortices. In this case, the complex velocity of the fluid $\mathrm{u} = u_x - iu_y$ is the meromorphic function

$$\mathrm{u}(z,t) = -i\Omega\bar{z} + i\sum_{i=1}^{N}\frac{\Gamma_i}{z - z_i(t)}, \tag{7}$$

where Ω is the angular velocity of the rotating fluid, N is the number of vortices, and Γ_i and $z_i(t)$ are circulations and positions of vortices.

Substituting this "pole ansatz" into Euler equation (6) shows that the number of vortices N and the circulations Γ_i do not change in time, while the moving positions of vortices $z_i(t)$ obey the Kirchhoff equations:

$$\dot{\bar{z}}_i = -i\Omega\bar{z}_i + i\sum_{i\neq j}^{N}\frac{\Gamma_j}{z_i(t) - z_j(t)}. \tag{8}$$

The Kirchhoff equations replace nonlinear partial differential equation (PDE) (6) by a dynamical system. They can be used for different purposes. The equations describe chaotic motions of a finite number of vortices if $N > 3$. If N is large, Kirchhoff equation can be used to approximate virtually any flow.

3.2. Chiral flow

The flows relevant for the FQHE are such that a large number of vortices largely compensates rotation. We refer to such flows as the chiral flow.

Bearing the quantum case in mind, we assume that vortices have the same (minimal) circulation $\Gamma_i = \Gamma$. Then the Kirchhoff equations become

$$\mathrm{v}_i \equiv \dot{\bar{z}}_i = -i\Omega\bar{z}_i + i\sum_{i\neq j}^{N}\frac{\Gamma}{z_i(t) - z_j(t)}. \tag{9}$$

We want to study the vortex system in the limit of a large number of vortices distributed with the mean density $\bar{\rho}$:

$$N \to \infty: \quad \bar{\rho} = \frac{\Omega}{\pi\Gamma}. \tag{10}$$

The chiral flow is a very special flow in fluid mechanics. We distinguish two types of motion there: the fast motion of the fluid around vortex cores and the slow motion of vortices. In this respect, vortices themselves can be considered a (secondary) fluid. In the ground state of the vortex fluid, vortices do not move, but the fluid does.

Circulation of vortices in units of the Planck constant has the dimension inverse to the mass unit. We introduce the dimensionless parameter

$$\nu = \frac{\hbar}{m_*\Gamma}. \tag{11}$$

We show in what follows that the quantized chiral flow models the FQHE with a filling fraction ν. We set $\beta = \nu^{-1}$.

3.3. Quantum Kirchhoff equations

Kirchhoff himself wrote Eqs. (9) in the Hamiltonian form, identifying the holomorphic and antiholomorphic coordinates of vortices as canonical variables. In the case of the rotating fluid the canonical variables are $m_*\Omega\bar{z}_i$ and z_i. The Hamiltonian of the chiral vortex system is given by

$$\mathcal{H} = m_*\Omega\left(\sum_i[\Omega|z_i|^2 - \Gamma\sum_{j\neq i}\log|z_i - z_j|^2\right), \tag{12}$$
$$(m_*\Omega)\{\bar{z}_i,\, z_j\}_{P.B.} = -i\delta_{ij}.$$

We emphasize that the Kirchhoff Hamiltonian is only a part of the energy of the fluid. This part of energy is transported by vortices. Another part of the energy is related to the vortices at rest. It diverges at vortex cores. This part is omitted in Eq. (12).

The parameter m_* introduced into the Hamiltonian and Poisson brackets sets the scale of energy. It is a phenomenological parameter that does not appear in the Kirchhoff equations.

The Kirchhoff vortex system is readily canonically quantized. We replace the Poisson brackets by the commutators

$$i\hbar\{\bar{z}_i,\, z_j\}_{P.B.} \to [\bar{z}_i,\, z_j] = 2\ell^2\delta_{ij}. \tag{13}$$

The parameter $2\ell^2 = \hbar/\Omega m_*$ has the dimension of area. It is a phenomenological parameter arising in quantization. We measure it in units of area per particle $2\ell^2 = \nu/\pi\bar{\rho}$. The dimensionless number ν in (11) is a semiclassical parameter. We see in what follows that ν is identified with the filling fraction and ℓ with the magnetic length.

The next step is the choice of states. We assume that states are holomorphic polynomials in z_i. Then the operators $\bar{z}_i$ are canonical momenta:

$$\bar{z}_i = 2\ell^2 \partial_{z_i}. \tag{14}$$

Finally, we have to specify the inner product. We impose the chiral condition: the operators $\bar{z}_i$ and z_i are Hermitian conjugate,

$$\bar{z}_i = z_i^\dagger. \tag{15}$$

This condition combined with representation (14) identifies the space of states with the Bargmann space [20] (see also [2]). This is the Hilbert space of analytic polynomials $\psi(z_1, \ldots, z_N)$ with the inner product

$$\begin{gathered} \langle \psi' | \psi \rangle = \int d\mu \, \bar{\psi}' \psi, \\ d\mu = \prod_i \exp\left(-\frac{|z_i|^2}{2\ell^2} \right) d^2 z_i. \end{gathered} \tag{16}$$

With Eqs. (9) and (14), we write the quantum velocity operators of vortices as

$$\begin{gathered} m_* \mathrm{v}_i = -2i\hbar \partial_{z_i} + i\hbar \sum_{i \neq j} \frac{\beta}{z_i - z_j}, \\ \dot{\bar{z}}_i = \mathrm{v}_i, \quad \beta = \nu^{-1}. \end{gathered} \tag{17}$$

We would like to emphasize a subtlety in quantizing velocities. Velocities are not the linear operators. They act on the phase of wave functions rather than on the wave function itself,

$$\mathrm{v}_i \exp\left(i \mathrm{Arg}\, \psi \right) = |\psi|^{-1} \left(-i\hbar \partial_{z_i} + i\hbar \sum_{j \neq i} \frac{\beta}{z_i - z_j} \right) \psi.$$

The linear operators are the momenta

$$\mathrm{p}_i = -i\hbar \left(\partial_{z_i} - \sum_{j \neq i} \frac{\beta}{z_i - z_j} \right). \tag{18}$$

Equations (9)–(18) are the quantum chiral Kirchhoff equations. They can be generalized to a sphere or a torus without difficulty.

4. QUANTUM CHIRAL KIRCHHOFF EQUATIONS AND THE FQHE

The quantum chiral Kirchhoff equations are readily identified with the FQHE.

The ground state of the vortex liquid is the state where the vortices are at rest. We repeat that this state is a highly excited state of the fluid. It is a state of the fluid at a very high angular moment. When vortices are in the ground state, the fluid moves with a very high energy.

The ground state is an analytic function whose phase is annihilated by all momenta operators. The common solution of the set of first-order PDEs

$$\mathrm{p}_i \psi_0 = 0$$

in the class of holomorphic polynomials is the Laughlin wave function in the Bargmann representation

$$\psi_0(z_1, \ldots, z_N) = \prod_{i>j} (z_i - z_j)^\beta, \quad \beta = 1/\nu. \tag{19}$$

The wave function is single-valued if β is a integer, antisymmetric if β is an odd integer, symmetric if β is an even integer.

The correspondence is completed when we assign the electronic charge to vortices and identify the angular velocity with the effective cyclotron frequency

$$\Omega = \frac{eB}{m_* c} = \frac{m_b}{m_*} \omega_c.$$

The hydrodynamic interpretation of the FQHE is subtly different from Laughlin's original interpretation. There, the coordinates entering the Laughlin wave function were interpreted as bare band electrons. The fluid itself is absent in the Laughlin picture. The hydrodynamic interpretation suggests that electrons (and their charge) are localized on topological excitations (vortices) of a neutral incompressible fluid. The neutral fluid is real. It serves as the agent of the interaction between electrons.

In the hydrodynamic interpretation, a quasihole [15] is a hole in the uniform background of vortices. It corresponds to state (2) characterized by a polynomial with simple zeros at a given point z,

$$Q(z_1, \ldots, z_N) = \prod_i^N (z - z_i). \tag{20}$$

The momentum of this state is

$$\mathrm{p}_i |Q\rangle = i\nu \frac{\Gamma}{z_i - z} |Q\rangle.$$

This shows that the Magnus force between vortices and the quasihole is the opposite to the fraction ν of the Magnus forces between vortices. Hence, in the hydrodynamic interpretation, the quasihole appears as a vortex with the fractional negative circulation $-\nu$.

Identifying vortices and electric charges, we must assume that the external fields (the potential well, gradients of temperature, etc.) are coupled to the vortices, not to the fluid.

We examine how vortices move in an external potential well $U(r)$. The potential adds the term $\sum_i U(r_i)$ to the Hamiltonian, where r_i are coordinates of vortices, and adds the force

$$-i[U, \bar{z}_i] = i2\ell^2 \partial_{z_i} U$$

to the Kirchhoff equations

$$\mathrm{p}_i = -i\hbar\partial_{z_i} + i\hbar \sum_{i\neq j} \frac{\beta}{z_i - z_j} + i m_* \ell^2 e \mathrm{E}(z_i), \qquad (21)$$

where $eE = -\nabla U$ is the electric field.

Fractionally quantized Hall conductance follows from the Kirchhoff equations easily. We assume that the electric field is uniform. Then the center of mass of the fluid stays at the origin, $\sum_i \partial_{z_i} = 0$. Summing (21) over all vortices, we obtain the Hall current per particle

$$N^{-1} \sum_i e\mathrm{v}_i = ie^2\ell^2 \mathrm{E}$$

and the current per volume $ie^2\ell^2\bar{\rho}\mathrm{E}$. We conclude that the Hall conductance equals to the fraction e^2/h:

$$\sigma_{xy} = \nu \frac{e^2}{h}. \qquad (22)$$

Our next step is to develop the hydrodynamics of a system of quantum vortices described by the Kirchhoff equations. To the best of our knowledge, this has not been done even for the classical fluids. We start by the summary of main results. The derivation and details then follow.

5. SUMMARY OF THE MAIN RESULTS AND DISCUSSION

Quantum hydrodynamics of a chiral vortex flow consists of three sets of data: the operator content and their algebra, the chiral constituency relation between operators, and the dynamic equation. We summarize them below, but first we comment on the notation.

5.1. Notation

We use holomorphic coordinates

$$z = x + iy, \quad \partial = \frac{1}{2}(\nabla_x - i\nabla_y).$$

We use the roman script for complex vectors. For example, the velocity of the fluid

$$u = (u_x, u_y), \quad \mathrm{u} = u_x - iu_y.$$

We denote the velocity of the vortex fluid

$$\mathrm{v} = v_x - iv_y,$$

the momentum flux for the vortex fluid

$$\mathrm{P} = P_x - iP_y,$$

and holomorphic components of symmetric flux tensors Π_{ab}:

$$\Pi = \Pi_{xx} - \Pi_{yy} - 2i\Pi_{xy}, \quad \Pi_{z\bar{z}} = \Pi_{xx} + \Pi_{yy}.$$

We emphasize the difference between Hermitian conjugation $\mathrm{v}^\dagger$ and complex conjugation $\bar{v}$, but still may use the classical notation for the divergence and the curl of the velocity. In particular, the divergence and the curl abbreviated as $\nabla \cdot v = 0$ actually means

$$\nabla \cdot v = \bar{\partial}\mathrm{v} + \partial\mathrm{v}^\dagger, \ \nabla \times v = i(\bar{\partial}\mathrm{v} - \partial\mathrm{v}^\dagger).$$

Similarly, the term $v \cdot \nabla\rho$ in (4) is understood as $\mathrm{v}^\dagger \cdot \partial\rho + \bar{\partial}\rho \cdot \mathrm{v}$.

The divergence-free velocity of an incompressible liquid is expressed in terms of the stream function operator

$$\mathrm{v} = -2i\partial\Psi. \qquad (23)$$

We define the momentum flux of the vortex flow as

$$\mathrm{P} = m_* \rho \mathrm{v}. \qquad (24)$$

The vortex flux operators annihilate the ground state:

$$\mathrm{P}|0\rangle = \langle 0|\mathrm{P}^\dagger = 0. \qquad (25)$$

Throughout the paper we set $m_* = 1$, measuring the momentum per particle in units of velocity, or equivalently, treating the particle density as a mass density. We emphasize that m_* is not related to the band electronic mass.

5.2. Commutation relation

Commutation relations of the vortex flux operators differ from the canonical commutation relations of quantum hydrodynamics of Landau [13] by the anomalous terms

$$\hbar^{-1}[\mathrm{P}(r), \mathrm{P}^\dagger(r')] = -\frac{1}{2}(P \times \nabla)\delta(r - r') + \underbrace{+ \frac{\hbar}{2\nu}\left(2\pi\rho^2\delta(r - r') + \frac{1}{4}\nabla\left[\rho \cdot \nabla\delta(r - r')\right]\right)}_{\text{anomalous term}}. \qquad (26)$$

The commutation relation between the flux and the density is canonical:

$$[\mathrm{P}(r), \rho(r')] = -i\hbar\rho\partial\delta(r - r'). \tag{27}$$

The vortex flux operator can be conveniently represented in terms of the canonical fields u and u†

$$\begin{aligned} \left[\mathrm{u}(r), \mathrm{u}^\dagger(r')\right] &= \pi\frac{\hbar^2}{\nu}\delta(r - r'), \\ [\mathrm{u}(r), \rho(r')] &= -i\hbar\partial\delta(r - r'). \end{aligned} \tag{28}$$

We introduce the axillary operator

$$\mathrm{J} = \rho\mathrm{u}, \tag{29}$$

which we call the vorticity flux. The hydrodynamic interpretation of this operator is to be given below. It has a canonical commutation relation with itself and with the density, but does not annihilate the vacuum. The vortex flux P does.

We show that the vortex flux and the vorticity flux differ by the anomalous term

$$\mathrm{P} = \mathrm{J} + i\frac{\hbar}{2\nu}\partial\rho. \tag{30}$$

The anomalous term adds to the diamagnetic energy of the flow in the background electromagnetic field,

$$\frac{e}{c}\int (A \cdot P)\, d^2r = \frac{e}{c}\int (A \cdot J) d^2r + \frac{\hbar}{4\nu}\frac{e}{c}\int B\,\rho d^2r,$$

effectively reducing the orbital moment of particles. Similarly, the anomalous term contributes to the angular momentum of the flow as

$$N^{-1}\int (r \times P)\, d^2r = N^{-1}\int (r \times J) d^2r + \frac{\hbar}{4\nu}.$$

The meaning of the anomalous term can be seen directly from the monodromy of FQH states (2). The monodromy with respect to a closed path is the phase acquired by the wave function when a particle is moved along that path. That is a circulation of the particle. It equals to the number of zeros of the wave function with respect to each coordinate. This number is $(n-1)/\nu$, where n is the number of particles enclosed by the path. It is less by ν^{-1} from the number of magnetic flux quanta piercing the system, simply because the vortex does not interfere with itself. The anomalous term accounts for that difference. The anomalous term can be regarded as a local version of the global relation between the monodromy of states and the number of particles. The difference, often called the shift $2\bar{s}$, has been emphasized in Ref. [22]. For the Laughlin states, $\bar{2}s = \nu^{-1}$.

5.3. Anomalous term in the chiral constituency relation

Unlike in a regular fluid mechanics, where the density ρ and velocity v are independent fields, they are related by the chiral constituency relation in the chiral flow. This means that the set of states on the lowest Landau level is restricted such that the velocity operator acts as a certain functional of the density operator.

In a very rough approximation, the chiral relation states that the vorticity per particle is the inverse filling factor in units of the Planck constant, as suggested in [6]. This view refers to a popular picture of the FQH states as electronic states with an additional amount of flux attached to each particle. The actual relation between the vorticity and the density is more complicated. It involves the anomalous term

$$\nabla \times v = \frac{h}{\nu}\left[\rho - \bar{\rho} + \underbrace{\frac{1}{4\pi}\left(\frac{1}{2} - \nu\right)\Delta\log\rho}_{\text{anomalous term}}\right], \tag{31}$$

where

$$\bar{\rho} = \nu(2\pi\ell^2)^{-1} = \nu\frac{e}{hc}B$$

is the mean density of electrons and $h = 2\pi\hbar$.

An accurate reading of this relation is: the action of the operators in the right-hand side and the left-hand side of (31) on the Bargmann "bra" state $\langle Q|$ are equal. They are not equal if the "bra" state is not in the Bargmann state.

In particular, a quasihole, a source for vorticity localized at r_0, corresponds to the polynomial $Q = \prod_i (z_0 - z_i)$. It deforms the density outside the core $r = r_0$ according to the equation[4)]

$$-\nu\delta(r - r_0) = \rho - \bar{\rho} + \frac{1}{4\pi}\left(\frac{1}{2} - \nu\right)\Delta\log\rho.$$

An equivalent form of the chiral relation connects the stream function and the density,

$$v_a = -\epsilon_{ab}\nabla_b\Psi, \quad \Psi = \frac{\hbar}{2\nu}\left[\varphi - \left(\frac{1}{2} - \nu\right)\log\rho\right], \tag{32}$$

where the "regular part" of the stream function φ is a solution of the Poisson equation

$$\Delta\varphi = -4\pi(\rho - \bar{\rho}). \tag{33}$$

4) Incidentally, a similar equation exists inside the vortex core. There, the quantum corrections change the last term to $-(1/4\pi)\nu\Delta\log\rho$. Accidentally, a similar equation followed from the effective action in Refs. [3, 4] erroneously featuring the term $-(1/4\pi)\nu\Delta\log\rho$ inside and outside the vortex.

We comment that the chiral relation readily extends to the case of an inhomogeneous magnetic field. In this case, the mean density $\bar{\rho} = \nu(e/hc)B$ in (31) and (33) is a function of coordinates. There are no other changes. In particular, in the ground state, where the velocity vanishes, the density in a nonuniform magnetic field obeys the "Liouville equation with a background". That is Eq. (31) with zero in the left-hand side. In the leading order in gradients, the ground-state density acquires the universal correction

$$\rho = \nu \frac{e}{hc} B - \frac{1}{4\pi}\left(\frac{1}{2} - \nu\right) \Delta \log B + \ldots \tag{34}$$

The integrated form of (31) is the sum rule connecting the angular momentum (per particle in units of $\hbar$)

$$\mathrm{L} = (\hbar N)^{-1} \int (r \times P)\, d^2 r$$

to the gyration per particle

$$N^{-1}\mathrm{G} = \int r^2 (\rho - \bar{\rho})\, d^2 r.$$

It is given by

$$\ell^2 \left[\mathrm{L} - \left(\frac{1}{\nu} - 2\right)\right] = \mathrm{G}. \tag{35}$$

The ground-state version of this formula is the familiar sum rule for the Laughlin wave function:

$$\nu \langle 0| \sum_i |z_i|^2 |0\rangle = \ell^2 N (N - 1 + 2\nu).$$

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5.4. Anomalous term in the Euler equation: Lorentz shear stress

Constituency relation (24), chiral condition (31), continuity equation (4), and the operator algebra in (26) and (27) constitute the full set of hydrodynamics equations for the chiral incompressible quantum fluid.

The chiral condition helps to write the continuity equation (4) as a nonlinear equation of the density alone:

$$\partial_t \rho - \frac{\hbar}{2\nu} \nabla\varphi \times \nabla\rho = 0, \quad \Delta\varphi = -4\pi(\rho - \bar{\rho}). \tag{36}$$

The equation is identical to the Euler equation for the vorticity in an incompressible fluid. Naturally, the anomalous term disappears from this equation. It appears in the boundary conditions, in the response to external fields, and also determines forces acting in the fluid.

Forces are rendered by the momentum flux tensor Π_{ab} entering the Euler equation, written in the form of the conservation law

$$\partial_t P_a + \nabla_\nu \Pi_{ab} = \rho F_a. \tag{37}$$

Here,

$$F = eE - \frac{e}{c} B \times v$$

is the Lorentz force.

The anomalous viscous stress emerges in the momentum stress tensor. A general fluid momentum flux tensor of incompressible fluid consists of the kinetic part, the stress, and the traceless viscous stress σ'_{ab}. In the incompressible fluid the stress is expected through the velocity. We write

$$\Pi_{ab} = \pi_{ab} - \sigma'_{ab}, \tag{38}$$

where π_{ab} accounts for the kinetic part and the stress. At the fixed density π_{ab} is symmetric with respect to a change of the direction of the velocity $v \to -v$. The viscous term is linear in gradients of the velocity. It changes the sign under this transformation. With the exception of the diamagnetic term, the viscous term has a lesser degree of velocity among terms of the flux tensor. This is the only term enters the linear response theory.

Our fluid is dissipation-free. Therefore, the anomalous viscous stress produces no work. This is possible if the viscous stress represents forces acting normally to a shear. Such stress can only be a traceless pseudotensor. It changes sign under the spatial reflection. In the chiral flow, the anomalous viscous stress is given by

$$\sigma'_{ab} = -\frac{\hbar}{2\nu}\rho \left(\nabla_a \nabla_\nu - \frac{1}{2}\delta_{ab}\Delta\right)\Psi. \tag{39}$$

There is a noticeable difference from the dissipative shear viscous stress. That stress is given by the same formula but with the stream function replaced by the hydrodynamic potential.

Components of the anomalous viscous stress tensor are

$$\begin{aligned} \sigma'_{xx} &= -\sigma'_{yy} = -\frac{\hbar}{4\nu}\rho(\nabla_x v_y + \nabla_y v_x), \\ \sigma'_{xy} &= \sigma'_{yx} = \frac{\hbar}{4\nu}\rho(\nabla_x v_x - \nabla_y v_y). \end{aligned} \tag{40}$$

The divergency of the Lorentz shear stress is the Lorentz shear force

$$\nabla_b \sigma'_{ab} = \frac{\hbar}{4\nu}\bar{\rho}\nabla_a(\nabla \times v)$$

exerted by the flow on the volume element of the liquid. It is proportional to the gradient of the vorticity. A notable feature of the anomalous stress is that the kinetic coefficient $1/4\nu$ (in units of $\hbar$) is universal and has a geometric origin. The anomalous conservative viscosity is referred to as the odd viscosity, or Hall viscosity. It was introduced in Ref. [8] for the integer Hall effect as a linear response to a shear. Its notion has been extended to the FQHE in [9, 10] (see [8–12] for incomplete set of references). In this paper, we show how the anomalous viscosity appears in the nonlinear hydrodynamics of the chiral flow.

Anomalous term (39) represents the force acting normally to the shear (in contrast, the shear viscous force acts in the direction parallel and opposite to the shear). The stress is also referred as the Lorentz stress, and the force is referred as the Lorentz shear force [9].

The emergence of the Lorentz shear stress can be interpreted in terms of semiclassical motion of electrons. The motion of electrons consists in the fast motion along small orbits and the slow motion of orbits. A shear flow strains orbits, elongating them normally to the shear, boundaries, and vortices. The elongation yields an additional Lorentz shear stress.

5.5. Topological sector

The topological sector consists of flows driven by slow long-wave external fields, such as the curvature of space, a nonuniform electric and magnetic fields, etc., which do not produce excitations over the gap. The Hall current is the most familiar example.

The topological sector can be singled out in the limit $m_* \to \infty$. In this limit, the momentum flux tensor reduces to the anomalous viscous stress modified by quantum corrections. Then the dynamics reduces to the balance between the Lorentz shear force and the Lorentz force.

In the linear approximation, the stationary Euler equation is

$$\left(\frac{1}{4\nu} - \frac{1}{2}\right) \nabla(\nabla \times v) = eE_a - \frac{e}{c} B \times v. \qquad (41)$$

Solution of this equation in the leading gradient approximation yields the universal correction to the Hall conductance [11]:

$$\frac{\sigma_{xy}(k)}{\sigma_{xy}} = 1 + \left(\frac{1}{4\nu} - \frac{1}{2}\right)(k\ell)^2, \quad \sigma_{xy} = \frac{\nu e^2}{h}. \qquad (42)$$

The Hall current increases with the wave vector. The factor $1/2$ in these equations represents the diamagnetic energy. This energy does not appear explicitly in the momentum flux tensor in (38). Rather, it is hidden in the normal ordering of the kinetic part of the vortex flux tensor. If in addition, particles prosses an orbital moment M, which is intrinsically related to the band, the term $(m_*/m_b)M$ is added to the factor $-1/2$ in both equations. Apart from this effect, the correction to the Hall conductance is universal.

5.6. Trace and mixed anomaly

The meaning of the Lorentz shear stress is best illustrated when the fluid is placed into a curved space. In this case, the energy receives an addition

$$H' = -\frac{1}{2} \int g^{ab} \sigma'_{ab} \sqrt{g} d^2 \xi$$

from the viscous tensor, where g_{ab} is the spatial metric. At a constant density, this term has the suggestive form

$$H' = \frac{\hbar}{4\nu} \bar{\rho} \int R \Psi \sqrt{g} d^2 \xi,$$

where R is the spatial curvature. This addition yields the trace anomaly: the flux tensor acquires a trace proportional to the curvature:

$$-\sigma'_{aa} = \bar{\rho} \frac{\hbar^2}{16\pi\nu} R. \qquad (43)$$

It is traceless if the space is flat.

The trace anomaly yields a uniform force acting toward the region with the access curvature. This force squeeze particles toward the curved regions (mixed anomaly)

$$\delta\rho = \frac{1}{8\pi} R \sqrt{g}. \qquad (44)$$

Accumulation of charges at curved parts of space was suggested in [22] and further discussed in [11].

These formulas represent the effect of the anomalous terms valid in the semiclassical approximation at large ν^{-1}. They experience quantum corrections, which effectively replace ν^{-1} in the formulas with $\nu^{-1} - 2$.

5.7. Dispersion of density modulation

The anomalous term in the commutation relations (26) yields a universal correction to the kinetic energy of small density modulations

$$|k\rangle = \sum_i e^{ikr_i} |0\rangle$$

(see footnote[3]) on page 3) of the form

$$\Delta_\nu(k) = \frac{1}{m_* \overline{\rho}^2}\langle k|\mathrm{P}(r)\mathrm{P}^\dagger(r)|k\rangle. \quad (45)$$

We will show that at small wave vectors the dispersion is negative

$$\Delta_\nu(k) = \Delta_\nu(0)\left(1 - \frac{1}{2}\left(\frac{1}{2\nu} - 1\right)(k\ell)^2\right),$$
$$\Delta_\nu(0) = \frac{\hbar^2}{2m_*\ell^2}. \quad (46)$$

Such behavior signals the magneto-roton minimum discussed in Ref. [2], similar to the roton minimum known in superfluid helium. The dispersion of the excitation has been measured in the recent work [23]. There the excitation spectrum has been probed through the resonant absorption in the regime where surface acoustic waves propagate across the sample.

5.8. Boundary double layer and dispersion of edge modes

A striking manifestation of the anomalous terms is seen on the boundary. The Lorentz shear force squeezes flow lines with different velocities. As a result the charge there is an accumulation of density on the adge. The density at the edge $r = R$ forms the double layer

$$\rho(r) = \overline{\rho} + \frac{1-\nu}{4\pi}\nabla_n \delta(r - R). \quad (47)$$

Here the derivative is taken in the direction normal to the boundary.

A consequence of the double layer is the correction to the spectrum of edge modes

$$\omega(k) = c_0 k + \frac{1}{2}\Delta_\nu\left(\frac{1}{\nu} - 1\right)\mathrm{sign}(k)(k\ell)^2,$$
$$c_0 = c\frac{E}{B}. \quad (48)$$

These results where obtained in [14].

In the rest of the paper, we obtain these (and some other) properties starting from the quantized chiral fluid. It turns out that many calculations are merely identical in the classical and quantum cases. To simplify the matter, we first derive the hydrodynamics of the vortex fluid in the classical case, and then consider the quantum case.

6. RELATION BETWEEN THE VORTEX FLOW VELOCITY AND THE FLUID VELOCITY

Eulerian hydrodynamics of the vortex flow describes the flow in terms of the density and the velocity field $v(r)$ of vortices. We construct the velocity field starting from velocities of individual vortices. The calculations are merely identical in the classical and quantum cases. We proceed with the classical calculations.

We denote density of vortices as

$$\rho(r) = \sum_i \delta(r - r_i) = \bar{\rho} + \frac{1}{2\pi\Gamma}(\nabla \times u). \quad (49)$$

The stream function of the fluid is the potential φ in (33):

$$\mathrm{u} = -2i\Gamma\partial\varphi, \quad \Delta\varphi = -4\pi(\rho - \bar{\rho}). \quad (50)$$

The object of interest is the vortex flux

$$\mathrm{P}(r) = \sum_i \delta(r - r_i)\mathrm{v}_i. \quad (51)$$

Having the flux, we define the velocity field of the vortex fluid as $\mathrm{P} = \rho\mathrm{v}$. We want to compute the velocity of the vortex flow $\mathrm{v}(r)$ and to compare it with the velocity of the original fluid $\mathrm{u}(r)$. Obviously, they are different. The former describes the slow motion of vortices, and the latter, the fast motion of the fluid around vortices and the drift together with the vortices. Nevertheless, there is a simple relation between the two.

We compute the vortex flux P and compare it with the vorticity flux $\mathrm{J} = \rho\mathrm{u}$, where the velocity of the fluid u is given by (7). Using (9) and the $\bar{\partial}$-formula $\pi\delta = \bar{\partial}(1/z)$, we write

$$\mathrm{P}(r) = \sum_i \delta(r - r_i)\left[-i\Omega\bar{z}_i + i\sum_{i,i\neq j}^{N}\frac{\Gamma}{z_i - z_j}\right] =$$
$$= -i\Omega\bar{z}\rho(r) + i\frac{\Gamma}{\pi}\bar{\partial}\sum_{i\neq j}^{N}\frac{1}{z - z_i}\frac{1}{z_i - z_j}. \quad (52)$$

Then use the identity

$$2\sum_{i\neq j}\frac{1}{z-z_i}\frac{1}{z_i-z_j} = \left(\sum_i \frac{1}{z-z_i}\right)^2 - \sum_i\left(\frac{1}{z-z_i}\right)^2 =$$
$$= \left(\sum_i \frac{1}{z - z_i}\right)^2 + \partial\sum_i\frac{1}{z - z_i} \quad (53)$$

and apply $\bar{\partial}$:

$$\rho\mathrm{v} = -i\Omega\bar{z}\rho + i\sum_i \delta(r - r_i)\sum_j \frac{\Gamma}{z - z_j} +$$
$$+ i\frac{\Gamma}{2}\partial\sum\delta(r - r_i). \quad (54)$$

We obtain the relations

$$\mathrm{P} = \rho \mathrm{u} + \frac{\Gamma}{2} i\partial\rho, \quad \mathrm{v} = \mathrm{u} + \frac{\Gamma}{2}\rho^{-1} i\partial\rho. \qquad (55)$$

The difference between the velocity of the vortex fluid and the velocity of the fluid has a simple meaning. The velocity of the fluid u diverges at the core of an isolated vortex (as can be seen in (7)). But the velocities of vortices are finite. The anomalous term removes that singularity.

The anomalous term changes only the transverse part of the velocity, and therefore the flow of vortices is incompressible like the fluid itself, $\nabla v = \nabla u = 0$. Also, the anomalous term does not change the divergency of the flux: $\nabla P = v\nabla\rho = \nabla(\rho u) = u\nabla\rho$.

7. CLASSICAL HYDRODYNAMICS OF THE VORTEX MATTER

Global symmetries of space and time, such as translation and rotation, yield familiar conservation laws of the flux, energy, and angular momentum. In addition, the $2D$ incompressible flows with a constant density possess conservation laws that are not directly related to global symmetries. One conservation law is familiar. This is the conservation of vorticity. With the help of (55), the Euler equation in form (6) can be written as the continuity equation for the mass density of the vortex fluid:

$$D_t\rho \equiv \left(\frac{\partial}{\partial t} + v\cdot\nabla\right)\rho = 0. \qquad (56)$$

In addition to the conservation of vorticity, the vorticity flux J and the vortex flux P are also conserved:

$$J = \rho u, \quad P = \rho v. \qquad (57)$$

The conservation of vorticity and vortex flux is obvious in the Kirchhoff picture. This is the conservation of mass and mass flux of the vortex system. In continuum fluid mechanics, conservation of the vorticity flux and, consequently, of the vortex flux are, perhaps, less obvious. Nevertheless, they easily follow from the observation that the vorticity flux is the divergence of a tensor:

$$J_a = \overline{\rho} u_a + \frac{1}{2\pi\Gamma}\epsilon_{ab}\partial_c t_{bc}, \quad t_{bc} = u_b u_c - \frac{1}{2}\delta_{bc}u^2. \qquad (58)$$

The tensor is symmetric and traceless.

We write the conservation law for the the vorticity flux

$$\partial_t J_a + \nabla_b \pi_{ab} = -\frac{e}{c}(B\times u)_a \qquad (59)$$

and determine the vorticity flux tensor π_{bc}. The right-hand side of this equation is the Lorentz force.

The vorticity flux tensor can be locally and explicitly expressed through the velosity and the pressure. Expression is cumbersome and we do not need in for the purpose of this paper. In the leadind approximation in the density gradients the second term in (56) could be dropped. Then the vorticity flux tensor is identical to the flux tensor of the incompressible fluid with the constant density

$$\pi_{ab} \approx \overline{\rho} u_a u_b + p\delta_{ab}.$$

The next step is to determine the vortex flux tensor Π_{ab}. It enters into the conservation law for the vortex flux

$$\partial_t P_a + \nabla_b \Pi_{ab} = -\frac{e}{c}(B\times v)_a. \qquad (60)$$

We see it as a transformation of the vorticity flux tensor induced by the transformation of the velocity (55)

$$u\to v, \quad J\to P, \quad \pi\to\Pi. \qquad (61)$$

Under the shift (55) we have

$$\dot{P}_a = \dot{J}_a + \frac{\Gamma}{4}\epsilon_{ab}\nabla_b\dot{\rho}.$$

With the help of the continuity equation (56), we obtain the transformation

$$\pi_{ab}\to\Pi_{ab} = \pi_{ab} + \frac{\Gamma}{4}\left[\epsilon_{ac}\nabla_c(\rho v_b) + \epsilon_{bc}\nabla_c(\rho v_a)\right].$$

In the leading approximation in gradients we replace the density in the last equation by its mean $\overline{\rho}$. We observe that the stress tensor acquires the anomalous viscous term

$$\Pi_{ab} \approx \pi_{ab} - \sigma'_{ab}, \qquad \sigma'_{ab} = -\frac{\Gamma}{4}\overline{\rho}\left(\epsilon_{ac}\nabla_c v_b + \epsilon_{bc}\nabla_a v_c\right). \qquad (62)$$

This is the Lorentz shear stress [14].

We see that the Lorentz shear stress naturally appears in the vortex liquid. Chiral flows consists of a fast motion along small orbits around vortex cores and a slow drift of centers of these orbits. A shear flow strains orbits elongating them normal to the shear, boundaries and vortices. Elongation yields to the Lorentz shear stress.

8. QUANTUM HYDRODYNAMICS OF THE VORTEX MATTER

We start by quantizing the incompressible chiral $2D$ fluid and then proceed with the quantization of the vortex flow.

8.1. Quantum hydrodynamics of incompressible liquid

The canonical fields in hydrodynamics are density and velocity. In the chiral fluid with a constant fluid density, the canonical hydrodynamic variables are the velocity u and the vorticity ρ, or rather, holomorphic and antiholomorphic components of the velocity u and $u^\dagger$.

We note a subtlety in quantizing hydrodynamics in the Bargmann space. The density in (49) is real and therefore consists of holomorphic and antiholomorphic variables. We "decompose" it into the holomorphic and antiholomorphic parts using the $\bar\partial$-formula

$$\pi\delta(r) = \bar\partial\left(\frac{1}{z}\right) = \partial\left(\frac{1}{\bar z}\right) \quad (63)$$

as

$$\rho(r) = \rho_+ + \rho_- = \frac{1}{2\pi}\bar\partial\sum_i\frac{1}{z-z_i} + \frac{1}{2\pi}\partial\sum_i\frac{1}{\bar z-\bar z_i}. \quad (64)$$

In the Bargmann space, the action of the holomorphic operator $2\partial_{z_i}$ on the density is not just a differentiation over coordinates $\partial_{x_i} - i\partial_{y_i}$, as it may seem from the notation. The operator acts only on the holomorphic part ρ_+. Hence, $2\partial_{z_i}\rho = -\partial\delta(r-r_i)$ is half the regular derivative. We already encountered this subtlety in Sec. 3.3 in discussing the action of velocity in the "first quantized" formalism.

With this nuance, the quantization of the fluid velocity amounts to the replacement of the term $-i\Omega\bar z$ in (7) with $\partial\pi_\rho$, where

$$\pi_\rho = -i\hbar\frac{\delta}{\delta\rho}$$

is the canonical momentum of the density. We also replace the sum in (7) with the integral,

$$\sum_i\frac{\Gamma}{z-z_i} \to \Gamma\int\frac{\rho(\xi)}{z-\xi}d^2\xi = = -i\frac{\hbar}{\nu}\partial(\varphi + \pi\bar\rho|z|^2). \quad (65)$$

We obtain the velocity of the quantum chiral fluid

$$\mathrm{u} = \partial\left(\pi_\rho - i\frac{\hbar}{\nu}(\varphi + \pi\bar\rho|z|^2)\right). \quad (66)$$

This formula yields the canonical commutation relation between vorticity and velocity and between the velocity components:

$$[\mathrm{u}(r), \rho(r')] = -i\hbar\partial\delta(r-r'),$$
$$\nabla\times u = i(\bar\partial\mathrm{u} - \partial\mathrm{u}^\dagger) = \frac{h}{\nu}(\rho - \bar\rho). \quad (67)$$

The commutation relations between velocity components are the canonical Heisenberg algebra, as is known to be the case in a quantizing magnetic field:

$$[\mathrm{u}(r), \mathrm{u}^\dagger(r')] = \frac{h^2}{\pi\nu}\delta(r-r'), \quad [\mathrm{u}(r), \mathrm{u}(r')] = 0. \quad (68)$$

The algebra is completed by the equal-point commutator

$$[\mathrm{u}(r), \rho(r)] = -i\hbar\partial\rho(r). \quad (69)$$

The remaining element of the quantization is the chiral condition. The holomorphic derivative ∂_{z_i} acting to the left on the antiholomorphic "bra" states of Bargmann space (2) differentiates only the factor $\exp\left(-\sum_i|z_i|^2/2\ell^2\right)$ of the measure, $\langle Q|\left(2\ell^2\partial_{z_i}^T + \bar z_i\right) = 0$. Similarly, the operator $\partial\pi_\rho$ acting on the left acts only on the factor $\exp\left(-(1/2\ell^2)\int\rho d^2r\right)$:

$$\langle Q|\left(\partial\pi_\rho + i\frac{\hbar}{\ell^2}\bar z\right) = 0. \quad (70)$$

Therefore, when the holomorphic velocity operator acts on the antiholomorphic "bra" state, the first two terms in (66) cancel. We return to the classical formula (50):

$$\langle Q|\mathrm{u} + i\frac{\hbar}{\nu}\partial\varphi|Q'\rangle = 0. \quad (71)$$

We emphasize that this relation does not hold unless the operator is sandwiched between antiholomorphic and holomorphic states.

The chiral condition projects all operators onto the lowest Landau level. The projected velocity is manifestly divergence-free. Projection onto the lowest Landau level is summarized by the condition $\Delta\pi_\rho = -4\pi\bar\rho$.

Heisenberg algebra of velocities (68), continuity equation for the vorticity $D_t\rho = 0$ (56), and chiral condition (71) summarize the quantization of hydrodynamics of an incompressible chiral flow.

Finally, we are ready to proceed with quantization of the vortex fluid.

8.2. Quantization of the vortex fluid

The classical formula for the flux, Eq. (51), must be treated as an ordered product of operators,

$$\mathrm{P}(r) = \sum_i\delta(r-r_i)\mathrm{p}_i = \sum_i(\mathrm{p}_i + i\hbar\partial_{z_i})\delta(r-r_i), \quad (72)$$

where the momenta p_i are given by (21). The relation between the velocity in (55) holds on the quantum level:

$$\mathrm{P} = \rho\mathrm{u} + i\frac{\hbar}{2\nu}\partial\rho. \quad (73)$$

The chiral condition is obtained by placing u to the left. Using (69), or equivalently (72), we pull u to the left and reduce it to its classical value (50). This yields the chiral conditions in Sec. 5.3:

$$\mathrm{P} = -i\frac{\hbar}{\nu}\rho\partial\varphi + i\hbar\left(\frac{1}{2\nu}-1\right)\partial\rho. \tag{74}$$

The commutation relations for flux components presented in Sec. 5.2, Eqs. (26) and (27), now follow.

The computation of the quantum vortex flux tensor is not much different from the classical version in Sec. 7. All the formulas remain the same if the normal ordering of operators is respected. But when the velocity in all terms of the vortex flux is pulled to the left, the coefficient in front of the Lorentz force acquires the quantum correction $1/2\nu \to 1/2\nu - 1$.

9. APPLICATIONS

9.1. Structure factor

Anomalous commutation relations allow computing the structure factor. This is

$$s_k = N^{-1}\langle 0|\rho_k\rho_{-k}|0\rangle,$$

where $\rho_k = \sum_i^N \exp(ikr_i)$ is the Fourier mode of a small density modulation with the wave vector k.

The chiral condition connects the density and flux modes. We evaluate it in the linear approximation in density modes. Using $k^2\varphi_k = 4\pi\rho_k$ in (74), we write the Fourier mode of the flux in terms of the density modes:

$$\mathrm{P}_k = \frac{\hbar \mathrm{k}}{(\ell k)^2}\left(1-\frac{1}{2}\left(\frac{1}{2\nu}-1\right)(k\ell)^2\right)\rho_k, \qquad \mathrm{k} = k_x - ik_y. \tag{75}$$

On the other hand, commutation relation (27) yields

$$[\mathrm{P}_k, \rho_{-k}] = \frac{1}{2}N\hbar\mathrm{k}. \tag{76}$$

Since $\mathrm{P_k}$ annihilates the ground state,

$$\langle 0|[\mathrm{P}_k,\, \rho_{-k}]|0\rangle = \langle 0|\mathrm{P}_k\rho_{-k}|0\rangle.$$

We obtain the relation

$$\langle 0|\mathrm{P}_k\rho_{-k}|0\rangle = \frac{\hbar\mathrm{k}}{(\ell k)^2}\left(1-\frac{1}{2}\left(\frac{1}{2\nu}-1\right)(k\ell)^2\right)\times \times\langle 0|\rho_k\rho_{-k}|0\rangle = \frac{1}{2}\hbar\mathrm{k}N. \tag{77}$$

The known result [2] for the spectral factor follows:

$$s_k = \langle 0|\rho_k\rho_{-k}|0\rangle \approx \approx \frac{1}{2}(k\ell)^2\left(1+\frac{1}{2}\left(\frac{1}{2\nu}-1\right)(k\ell)^2\right)+\ldots \tag{78}$$

We see that the anomalous term accounts for the universal $\mathcal{O}(k^4)$ in the structure factor. The spectral factor is involved in a number of important physical objects. A few are discussed below.

9.2. Variational excitation spectrum

In this section we evaluate the variational energy of waves. That is the energy per particle of a state with the density modulation with the wave vector k:

$$\Delta(k) = \frac{1}{m_*\bar\rho^2}\langle 0|\mathrm{P}_k\,\mathrm{P}_k{}^\dagger|0\rangle$$

(in this subsection, we restore the inertia m_*). Commutation relation (26) prompts the relation between the kinetic energy of small density modulations and the structure factor. We take the vacuum expectation value of the anomalous commutation relations (26) and express it through the Fourier modes. The right-hand side becomes $\langle 0|\mathrm{P}_k\,\mathrm{P}_k{}^\dagger|0\rangle$. Computing the expectation value of the term ρ^2 in the right-hand side of (26), we use the quantum version of relation (49),

$$\rho = \bar\rho + i\frac{\nu}{\hbar}(\bar\partial\mathrm{u} - \partial\mathrm{u}^\dagger),$$

and pull the holomorphic (antiholomorphic) velocity component to the left (right) with the help of (69) and apply the chiral condition. We obtain

$$\rho^2 = \bar\rho^2 + \frac{\nu}{4\pi}\Delta\rho.$$

This term (the quantum correction) effectively shifts the coefficient in front of the last term of commutation relation (69). We obtain

$$\Delta(k) = \Delta(0)\left(1-\frac{1}{2}\left(\frac{1}{2\nu}-1\right)(k\ell)^2\right), \qquad \Delta(0) = \frac{\hbar^2}{2m_*\ell^2}. \tag{79}$$

The comparison with (78) yields a variational Feynman–Bijl formula [7] for the excitation spectrum

$$\Delta(k) = \frac{\hbar^2 k^2}{2m_* s_k}. \tag{80}$$

Of course, all these formulas make sense in the leading order in $(k\ell)^2$.

We observe that the excitation spectrum is gapped and has a negative dispersion. The energy starts to increase at larger $(k\ell)^2$. It oscillates at intermediate wavelengths. Such behavior signals the magneto-roton minimum, similar to the roton minimum known in superfluid helium, as it has been suggested in Ref. [2]. The dispersion of the excitation was recently measured in [23]. There, the excitation spectrum was probed through the resonant absorption in the regime where surface acoustic waves propagate across the sample.

To avoid possible confusion, we emphasize that we evaluated the kinetic energy over the states $|k\rangle = \sum_i^N \exp(ikr_i)|0\rangle$. These states are different from the "projected waves" in Ref. [2]. Projected plane waves are created by the normal-ordered wave operator

$$\sum_i \exp(-i\mathrm{k}\ell^2\partial_{z_i}) \exp\left(-i\frac{\bar{\mathrm{k}}}{2}z_i\right)|0\rangle.$$

Operators expanded in that basis are a separate interesting question. We will address it elsewhere. Here, we comment that the acoustic waves used in the experiment in [23] are argued to be projected plane waves. Rather, they are regular waves $|k\rangle = \sum_i^N \exp(ikr_i)|0\rangle$.

9.3. Hall conductance in a nonuniform background

The formulas in the previous section are readily adapted to study transport in the topological sector, e. g., in a nonuniform electric field.

An electric field acts only on vortices as the Lorentz force in (21). We therefore add it to the conservation law for the vortex flux:

$$\partial_t \mathrm{P}_a + \nabla_b \Pi_{ab} = \rho\left(eE - \frac{e}{c}B\times v\right)_a. \tag{81}$$

In the topological sector ($m_* \to \infty$), the flow is steady, and the anomalous viscous tensor is the only term of the flux tensor that survives in the limit:

$$-\nabla_b \sigma'_{ab} = \rho\left(eE - \frac{e}{c}B\times v\right)_a. \tag{82}$$

Pulling the velocity to the left, in the linear approximation, we obtain Eq. (41) in Sec. 5.5. That equation yields a universal correction to Hall conductance (42).

Comparing the expressions for spectral function (78) and the Hall conductance, we observe a simple relation between the two objects. It can be obtained from the general theory of linear response.

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