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EFFECT OF VORTEX PINNING BY POINT DEFECTS ON THE LOWER CRITICAL FIELD IN LAYERED SUPERCONDUCTORS

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The lower critical field H_{c1} in layered superconductors is calculated under the assumption that vortex pinning by point defects is strong in these materials. We consider the case of a purely electromagnetic coupling of vortex pancakes and the case of both the electromagnetic and Josephson couplings of the pancakes in a vortex line. In the latter case, singularities in the temperature dependence of H_{c1} are predicted at certain characteristic temperatures.

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1. INTRODUCTION

Effects of thermal fluctuations of vortices on the lower critical field H_{c1} and on the magnetization of type-II superconductors were considered in a number of papers [1–6]. It was shown that the fluctuations lead to a renormalization of the temperature dependence of H_{c1}. In addition, effects of flux-line pinning on the equilibrium magnetization M of superconductors were analyzed for the cases of pinning by point [7, 8] and columnar [9, 10] defects. In this paper, we consider the effect of pinning by point defects on the lower critical field in layered superconductors, leaving aside the analysis of this effect for three-dimensional superconducting materials.

In layered superconductors like $Bi_2Sr_2CaCu_2O_{8+\delta}$, a vortex is the stack of vortex pancakes (VPs) localized in superconductive layers, and the vortex elasticity ε_l displays two features that, as we see in what follows, result in a noticeable effect of vortex pinning by point defects on H_{c1}. Both these features are caused by large anisotropy of these superconductors. The first feature is that the elasticity is relatively small, and this smallness leads to the Larkin length L_c that does not exceed the interlayer spacing d. In other words, the characteristic pinning energy of a vortex pancake is larger than its characteristic elastic energy, and hence pinning of the VPs is strong in these superconductors at least for not too high temperatures T [11–13]. The second feature is that in contrast to the practically constant ε_l in three-dimensional superconductors, the elasticity in layered superconductors essentially depends on the scale of the vortex distortion, i. e., on the wave vector k_z along the vortex [11–14]. This function $\varepsilon_l(k_z)$ results from an interplay of the electromagnetic and Josephson couplings of the VPs in a vortex line.

In the experiments in [15, 16], the temperature dependences of the magnetization M were measured at various magnetic inductions B in $Bi_2Sr_2CaCu_2O_{8+\delta}$ crystals, and a second-order phase transition line $B_g(T)$ was observed in the vortex system of these superconductors at moderate temperatures of the order of 40 K. Since a second-order phase transition line cannot have a critical point similar to that of the first-order phase transition line between a liquid and its vapor [17], the end of the curve $B_g(T)$ in the T–B plane should lie on the line $B = 0$ that corresponds to the lower critical field. Hence, the experimental data in [15, 16] indirectly suggest that the Bi-based superconductors may exhibit a singular behavior of $H_{c1}(T)$ near a temperature close to 40 K.

This paper is organized as follows. In Sec. 2, a simple model for the vortex elasticity $\varepsilon_l(k_z)$ in layered superconductors is formulated, and in Sec. 3 the main formulas describing strong pinning of the VPs are presented. In Sec. 4, the field H_{c1} is studied in the case where purely electromagnetic coupling of the VPs in a vortex line occurs. In this situation, $H_{c1}(T)$ is renormalized both by thermal fluctuations of the vortex pan-

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cakes and by their pinning. The lower critical field in the case of both the Josephson and electromagnetic couplings of the VPs is considered in Sec. 5. In this case, the renormalization of $H_{c1}(T)$ is accompanied by singularities in the T-dependence of H_{c1} at certain temperatures. The results of the paper are briefly summarized in Sec. 6.

2. ELASTICITY OF A VORTEX LINE

In a layered superconductor, pinning forces and thermal fluctuations shift the VPs comprising a vortex line away from its axis, and the line is distorted. Below, we deal with the distortions with large wave vectors k_z of the order of π/d, where d is the interlayer spacing. For such k_z, the elasticity $\varepsilon_l(k_z)$ of a vortex line in a layered superconductor has the form [11–14]

$$\varepsilon_l(k_z) \approx \varepsilon_0 \left[\varepsilon^2 \ln\left(\frac{d}{\varepsilon u}\right) + \frac{1}{\lambda^2 k_z^2}\ln\left(\frac{\lambda}{u}\right)\right], \quad (1)$$

where $\varepsilon_0 = (\Phi_0/4\pi\lambda)^2$, λ is the planar London penetration depth, Φ_0 is the flux quantum, $\varepsilon \ll 1$ is the anisotropy parameter of the superconductor, and u is the amplitude of the vortex–pancake displacements. It is taken into account in Eq. (1) that in the case of strong pinning, the displacement u can be large, $uk_z > 1$. The first term in formula (1) describes the Josephson coupling of the VPs, and the second term is due to their electromagnetic interaction. The parameter $\varepsilon\lambda k_z$ characterizes the relative roles of the Josephson and electromagnetic couplings of the VPs in the elasticity of the vortex line.

The logarithmic factors in formula (1) are of the same order of magnitude when $\lambda \sim d/\varepsilon$. This situation just occurs in Bi-based superconductors at not too high temperatures (e.g., at $\lambda = 0.2\mu$m, $d = 1.5$ nm, and $\varepsilon = 1/200$, we obtain $\varepsilon\lambda/d \approx 0.7$). Hereafter, we replace the logarithmic factors by the quantity $q \equiv 0.5\ln(\kappa^2\xi^2/\langle u^2\rangle)$, where $\kappa = \lambda/\xi$ is the Ginzburg–Landau parameter, ξ is the planar coherence length, and $\langle u^2\rangle$ gives the averaged value of u^2 for the VPs in the case of strong pinning. The explicit value of this quantity q is given below (see formula (23)). To simplify our analysis further, we use the following model dependence for $\varepsilon_l(k_z)$ that reproduces the main features of Eq. (1):

$$\varepsilon_l(k_z) = \varepsilon_0 q \varepsilon^2, \quad k_z^{max} \ge k_z \ge k_\lambda, \quad (2)$$

$$\varepsilon_l(k_z) = \frac{\varepsilon_0 q}{\lambda^2 k_z^2}, \quad k_z \le k_\lambda. \quad (3)$$

This model is similar to that used in Refs. [18, 19] (in those papers, $q = 1$). Here, $k_z^{max} = \pi/d$ is the maximum value of k_z, and $k_\lambda \equiv (\varepsilon\lambda)^{-1}$. Formula (2) describes the Josephson coupling of the VPs, and Eq. (3) corresponds to their electromagnetic coupling.

To characterize the type of the coupling in a vortex, we define the parameter p as

$$p \equiv \frac{k_z^{max}}{k_\lambda} = \frac{\pi\varepsilon\lambda}{d}. \quad (4)$$

When $p < 1$, the region of the Josephson coupling is absent for all k_z. In this case, the elastic energy of a vortex

$$E_{el} = \int_0^{\pi/d} \frac{dk_z}{2\pi}\varepsilon_l(k_z)k_z^2|u(k_z)|^2 \quad (5)$$

can be represented in the form [12]

$$E_{el} = \sum_i E_{em} q \frac{u_i^2}{\xi^2}, \quad (6)$$

where u_i is the displacement of the VP in the ith layer of the superconductor,

$$u(k_z) = d\sum_i u_i \exp(-ik_z z_i)$$

is the corresponding Fourier transform, and $E_{em} \equiv \varepsilon_0 d\xi^2/\lambda^2$. Formula (6) shows that the VPs in different layers can be regarded as independent "particles" in an effective mean-field harmonic potential generated by all other VPs of the vortex line [12].

When $p > 1$, the elastic energy E_{el} consists of two parts, $E_{el} = E_{el}^{>} + E_{el}^{<}$. The Josephson coupling of the VPs comprising the vortex line occurs for the vibration modes of the vortex with k_z in the interval $k_z^{max} > k_z > k_\lambda$, and the elastic energy of these modes is

$$E_{el}^{>} = \varepsilon_0 q \varepsilon^2 \int_{k_\lambda}^{k_z^{max}} \frac{dk_z}{2\pi} k_z^2 |u(k_z)|^2. \quad (7)$$

On the other hand, the vibrating modes with $k_z < k_\lambda$ lead to an uncorrelated motion of vortex segments of the length $L_\lambda = (k_z^{max}/k_\lambda)\, d = \pi\varepsilon\lambda$, and the elastic energy of these longwave modes is given by an expression similar to Eq. (6),

$$E_{el}^{<} = \sum_j E_{em} q \frac{L_\lambda}{d}\frac{\bar{u}_j^2}{\xi^2}, \quad (8)$$

where $\bar{u}_j$ is the displacement of the jth segment as a whole.

In Secs. 3 and 4, the case of purely electromagnetic coupling of the VPs ($p < 1$) is considered, whereas the case $p > 1$ is discussed in Sec. 5.

3. STRONG PINNING OF THE VORTEX PANCAKES

Strong pinning of the VPs was analyzed in Refs. [11–13]. Here, using somewhat different approach, we derive the appropriate formulas again and present them in the form that permits us to use the obtained equations at realistic values of the vortex elasticity and pinning.

We consider an individual VP in a pinning potential generated by point defects. The distribution $w(E)$ of its potential energies is Gaussian[1)] [11]:

$$w(E) = \frac{1}{\sqrt{\pi}U_p} \exp\left(-\frac{E^2}{U_p^2}\right), \qquad (9)$$

where the parameter U_p is of the order of $U_p^0 = \xi(f_p^2 n_p \xi^2 d)^{1/2}$, the characteristic pinning energy of the VPs; f_p is the mean pinning force caused by a point pinning center; and n_p is the density of these centers. For low B and T, we have $U_p^0, U_p \gg E_{em}$ for Bi-based superconductors [11–13]. As in Refs. [11–13], we assume that for the unit area of a superconducting layer containing the vortex pancake, the number of the pinning-potential extrema lying below an energy E is given by

$$n(E) = \frac{1}{\pi\xi^2} \int\limits_{-\infty}^{E} dE' w(E') = \frac{1 + \mathrm{erf}(E/U_p)}{2\pi\xi^2}, \qquad (10)$$

where $1/\pi\xi^2$ is the density of these extrema, i. e., of pinning wells and humps, and $\mathrm{erf}(x)$ is the probability integral [20],

$$\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int\limits_0^x dt\, \exp(-t^2). \qquad (11)$$

We now consider a VP in the vortex line. Its total energy is the sum of its energy in the pinning potential and of its elastic energy. The pinning potential "stimulates" the pancake to seek the deepest minimum of this potential in the appropriate layer. On the other hand, the displacement u of the vortex pancake from the vortex-line axis leads to an increase in its elastic energy $E_{el}(u) = E_{em} q u^2/\xi^2$. At $T = 0$, in each layer, the appropriate vortex pancake occupies the energy minimum with the lowest total energy, i. e., the absolute energy minimum in the layer. To proceed with the analysis of this absolute minimum, we first estimate the distribution of the local energy minima in the layer in the case of strong collective pinning of the VPs by point defects. This strong pinning occurs when the characteristic scale of the pinning potential, U_p, is essentially larger than the characteristic elastic energy $E_{em}q$, i. e., when

$$\delta \equiv \frac{U_p}{qE_{em}} \gg 1. \qquad (12)$$

In this case, any of the VPs forming the vortex line can "explore" many wells of the pinning potential, and its total energy has many local minima in the layer. The number $g_m(E)\, dE$ of these minima in the interval from some $E < 0$ to $E + dE$ is obtained as

$$g_m(E) = \int\limits_0^\infty 2\pi u\, du \frac{dn(E - E_{el}(u))}{dE} = \int\limits_0^\infty \frac{dE_{el}}{E_{em}q} w(E - E_{el}) = \frac{1}{E_{em}q} \int\limits_{-\infty}^{E} d\epsilon\, w(\epsilon) = \frac{\pi\xi^2 n(E)}{E_{em}q} = \frac{1 + \mathrm{erf}(E/U_p)}{2E_{em}q}, \qquad (13)$$

where $2\pi u du \cdot dn(E - E_{el}(u))$ is the number of the minima in the infinitesimal ring bounded by u and $u + du$, and we have changed the integration variable from u to E_{el}. With the function $g_m(E)$, the condition

$$\int\limits_{-\infty}^{E_0} g_m(E)\, dE = 1 \qquad (14)$$

determines $E_0 < 0$, the upper boundary of the energies of the VPs forming the vortex line at $T = 0$. Condition (14) means that $g_m(E)$ at $E \le E_0$ is the probability density for a vortex pancake inside the vortex line to be in the absolute energy minimum E.

With formula (13), Eq. (14) for E_0 can be rewritten in the form

$$\frac{\delta}{2}\left[x_0[1 + \mathrm{erf}(x_0)] + \frac{1}{\sqrt{\pi}} \exp(-x_0^2)\right] = 1, \qquad (15)$$

where $x_0 \equiv E_0/U_p < 0$. When the parameter δ is so large that $|x_0| \gg 1$, Eq. (15) reduces to

$$\frac{U_p}{4E_{em}q\sqrt{\pi}\, x_0^2} \exp(-x_0^2) \approx 1, \qquad (16)$$

and its approximate solution is

$$E_0 = U_p x_0 \approx -U_p \left[\ln\left(\frac{U_p}{4E_{em}q\sqrt{\pi}}\right)\right]^{1/2}. \qquad (17)$$

1) A uniform distribution of point defects leads to a renormalization of λ and hence of H_{c1}. This renormalization of H_{c1} is proportional to the mean density of the defects, n_p, and is not considered here. The pinning potential is generated by spatial fluctuations of the density around n_p, and hence the mean energy for distribution (9) is zero.

In obtaining Eq. (16), the following expression for $\mathrm{erf}(x)$ in the limit $x \gg 1$ has been used [20]:

$$\mathrm{erf}(x) \approx 1 - \frac{1}{\sqrt{\pi}\,x}\exp(-x^2)\left(1 - \frac{1}{2x^2}\right). \quad (18)$$

The VP energy averaged over the layers is the pinning energy of the pancake in the vortex line,

$$E_{pin} = \int_{-\infty}^{E_0} E g_m(E)\, dE. \quad (19)$$

Using Eq. (13), we arrive at

$$E_{pin} = \frac{U_p^2}{4E_{em}q} \times \times \left[(x_0^2 - 0.5)[1 + \mathrm{erf}(x_0)] + \frac{x_0}{\sqrt{\pi}}\exp(-x_0^2)\right], \quad (20)$$

where $x_0 = E_0/U_p$. Taking Eq. (15) into account, the energy E_{pin} can also be rewritten in the form

$$E_{pin} = = E_0\left(\frac{1}{2} - \frac{1}{4x_0^2} + \frac{U_p}{8\sqrt{\pi}E_{em}qx_0^2}\exp(-x_0^2)\right). \quad (21)$$

This expression together with formula (16) reveals that E_{pin} tends to E_0 when $|x_0| \gg 1$. In Fig. 1, the energies $|E_{pin}|$ and $|E_0|$ are shown as functions of the parameter δ. It can be seen that in the interval $100 > \delta > 20$, the energy $|E_{pin}|$ is approximately 20–40 % larger than $|E_0|$.

We next calculate $\langle u^2\rangle$, the averaged shift of the VPs forming the vortex line from the axis of this line,

$$\langle u^2\rangle = \int_{-\infty}^{E_0} dE \int_0^{\infty} 2\pi u^3 du \frac{dn(E - E_{el}(u))}{dE} = = \frac{\pi\xi^4}{(E_{em}q)^2}\int_{-\infty}^{E_0}(E_0 - E')n(E')\, dE' = = \frac{\xi^2(E_0 - E_{pin})}{E_{em}q}. \quad (22)$$

In the limit $|x_0| = |E_0|/U_p \gg 1$, Eq. (22) gives

$$\frac{\langle u^2\rangle}{\xi^2} \approx \frac{U_p}{2qE_{em}}\left[\ln\left(\frac{U_p}{4E_{em}q\sqrt{\pi}}\right)\right]^{-1/2}. \quad (23)$$

Omitting all logarithmic factors under the sign of the logarithm, we find the following estimate of the quantity $q = 0.5\ln(\kappa^2\xi^2/\langle u^2\rangle)$ introduced in Sec. 2: $q \approx \approx 0.5\ln(\kappa^2 E_{em}/U_p)$.

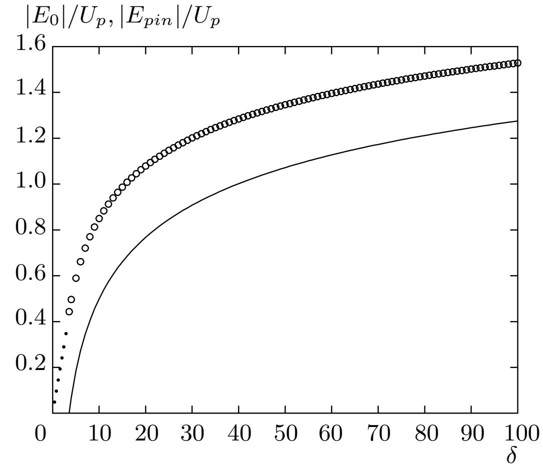

Fig. 1. Energies $|E_0|$ (solid line), Eq. (15), and $|E_{pin}|$ (circles), Eq. (20), as functions of the parameter $\delta = U_p/(E_{em}q)$. The energy $|E_{pin}|$ in the case of weak pinning, Eq. (24), is shown by dots. All the energies are measured in units of U_p

Formulas (17) and (23) agree with the appropriate results obtained in Refs. [11–13], where strong pinning of the VPs was analyzed in the limit $|E_0| \gg U_p$. However, for realistic values $\delta \lesssim 100$, the limit $|E_0| \gg U_p$ is not reached (see Fig. 1), and hence expressions (13)–(15) and (19)–(22) for $g_m(E)$, E_0, E_{pin}, and $\langle u^2\rangle$ permit us to find these quantities in realistic situations.

Moreover, expressions (13)–(15) and (19)–(22) also allow extrapolating the quantities $g_m(E)$, E_0, E_{pin}, and $\langle u^2\rangle$ from the region $\delta \gg 1$ to the boundary ($\delta \sim \sim 1$) between the regimes of strong and weak pinning. Here, we estimate this boundary as the point at which E_0 reaches zero. According to Eq. (15), this occurs at $\delta = 2\sqrt{\pi}$, and at this point, $E_{pin} = -U_p^2/(8E_{em}q) = = -U_p\sqrt{\pi}/4 \approx -0.44U_p$ and $\langle u^2\rangle/\xi^2 = \pi/2$. Of course, these values are only estimates because the derivation of g_m given in Eqs. (13) fails at $\delta \sim 1$, and at such δ, the exact $g_m(E)$ would generally differ from the expression used here.

For completeness, we present a formula for E_{pin} in the case of weak pinning of the VPs. In this case, the VP displacement u is found from the balance between the mean pinning force U_p^0/ξ and the elastic force $2E_{em}qu/\xi^2$ [i. e., $u/\xi = U_p^0/(2E_{em}q)$], and the pinning energy of the pancake is

$$E_{pin} = E_{em}q\left(\frac{u}{\xi}\right)^2 - \frac{U_p^0}{\xi}u = -\frac{(U_p^0)^2}{4E_{em}q}. \quad (24)$$

At the boundary of the weak pinning regime, u reaches ξ, i. e., we have $U_p^0 = 2E_{em}q$ and $E_{pin} = -E_{em}q$. If we impose the requirement that Eq. (20) gives the same energy $-E_{em}q$ at this boundary, we find that this occurs at $\delta \approx 2.88$ and $E_{pin} \approx -0.35U_p$. We note that this boundary $\delta \approx 2.88$ is relatively close to the value $2\sqrt{\pi} \approx 3.54$ estimated above from the side of strong pinning (see Fig. 1).

4. FREE ENERGY OF A VORTEX LINE WITH PURELY ELECTROMAGNETIC COUPLING OF THE VORTEX PANCAKES

4.1. General formulas

At $p < 1$ (the case of a purely electromagnetic coupling), positions of the VPs comprising the vortex line in different superconducting layers are not correlated (Sec. 2). Let $E \le E_0$ be the minimum energy of a VP in one of the layers. Then the free energy of this pancake can be written in the form

$$f_{pnc} = e_0 d + E - T\ln Z(E), \tag{25}$$

where $e_0 = \varepsilon_0 \ln\kappa$ is the usual expression for the energy of a vortex per its unit length,

$$Z(E) = \int_E^{\infty} g(E')\exp\left(-\frac{E'-E}{T}\right) dE' \tag{26}$$

is the partition function of the VP, and $g(E)$ is the density of VP states in the elastic and pinning potentials. The last (entropy) term in Eq. (25) is caused by thermal fluctuations of the VPs, and this term takes into account that at $T > 0$, the pancake can occupy not only its optimal energy state.

The lower critical field $H_{c1} = 4\pi f/\Phi_0$ is determined by the free energy f of a vortex per its unit length. Averaging expression (25) over the layers with the function $g_m(E)$, this free energy f can be represented as

$$f = e_0 + \frac{1}{d}E_{pin} - \frac{T}{d}\ln Z, \tag{27}$$

where $E_{pin} < 0$ is defined by Eq. (19) and

$$\ln Z = \int_{-\infty}^{E_0} \ln[Z(E)]\, g_m(E)\, dE. \tag{28}$$

In distinction to $g_m(E)$ describing the distribution of the energy minima of VPs in a vortex line, $g(E)$ gives the total density of states for such VPs, including the states in which the pinning and the elastic forces acting on the pancakes are not balanced. As the starting point, we consider the density of states $g(E)$ in the case where the pinning of the VPs is absent, i. e., when the VPs are in the elastic potential only, $E = E_{el}(u) = E_{em}qu^2/\xi^2$. In this case, we have $g(E) = g_{el}(E)$, where

$$\begin{aligned} &g_{el}(E) = 0, \quad E < 0,\\ &g_{el}(E) = \frac{2\pi u\, du}{s_0 dE} = \frac{\pi\xi^2}{s_0}\frac{1}{E_{em}q}, \quad E > 0, \end{aligned} \tag{29}$$

$2\pi u\, du$ is the area of the infinitesimal ring from u to $u + du$, and the elemental area s_0 determines the number $1/s_0$ of states of an individual vortex pancake per unit area. It was assumed in [4] that this area is of the order of $\pi\xi^2$, while in [6], s_0 was found from an analysis of the superconducting order-parameter excitations in the vortex core. In analyzing the effect of pinning on H_{c1}, the exact value of s_0 is not important, and we do not fix it here.

Interestingly, $g_{el}(E)$ can also be obtained from formulas (13) for g_m if we multiply this g_m by the factor $\pi\xi^2/s_0$ and set $U_p = 0$. Indeed, in this case, $w(E)$ in Eq. (9) becomes the delta function, $w(E) = \delta(E)$, and formula (13) transforms into

$$g_{el}(E > 0) = \frac{\pi\xi^2}{s_0}\frac{1}{E_{em}q}\int_{-\infty}^{E} d\epsilon\, w(\epsilon) = \frac{\pi\xi^2}{s_0}\frac{1}{E_{em}q}.$$

Generalizing this property of $g_{el}(E)$ to the case where the VP experiences both the elastic and pinning potentials, we assume below that $g(E)$ is given by the relation $g(E) = (\pi\xi^2/s_0)g_m(E)$, i. e.,

$$g(E) = \frac{\pi\xi^2}{s_0}\frac{1}{E_{em}q}\int_{-\infty}^{E} d\epsilon\, w(\epsilon) = \frac{\pi\xi^2}{s_0}\frac{[1+\mathrm{erf}(x)]}{2E_{em}q}, \tag{30}$$

where $x \equiv E/U_p$. Formula (30) shows that pinning smoothes the sharp step that occurs in $g_{el}(E)$ in the absence of the pinning potential, and the scale of this smoothing is U_p, as we see in Fig. 2. Thus, our assumption is no more than a simple realization of the quite natural idea on the effect of pinning on $g(E)$.

4.2. Analysis of the formulas

When pinning of the VPs is absent ($E_0 = E_{pin} = 0$), the partition function is simple,

$$Z = \int_0^{\infty} g_{el}\exp\left(-\frac{E'}{T}\right) dE' = Tg_{el}, \tag{31}$$

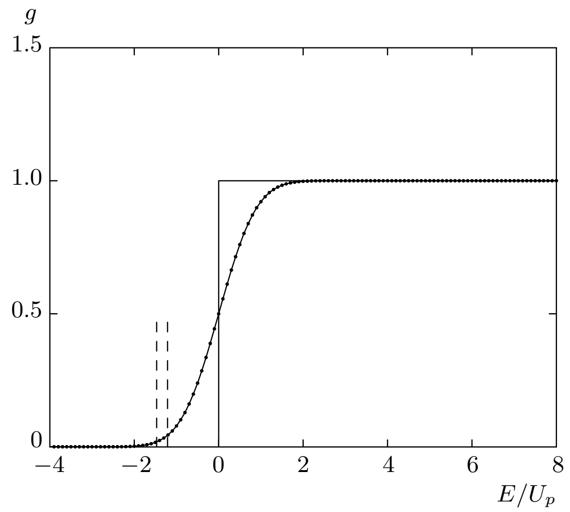

Fig. 2. The density of states $g(E)$ of a vortex pancake, Eq. (30), as a function of its energy E (solid line with dots). For comparison, the solid line shows the function $g_{el}(E)$, Eq. (29). Both these functions are measured in units of $(\pi\xi^2)/(s_0E_{em}q)$, whereas E is in units of U_p. The dashed lines mark the energies $E_0 = -1.21U_p$ and $E_{pin} = -1.47U_p$ calculated at $\delta = 80$

with the constant $g_{el} = g_{el}(E > 0)$, Eq. (29). Then the contribution of the thermal fluctuation of the vortex pancakes to the free energy is given by

$$f_T = -\frac{T}{d}\ln(Tg_{el}), \tag{32}$$

and the lower critical field H_{c1}^T renormalized by these thermal fluctuations takes the form

$$H_{c1}^T = H_{c1}^0(T) - \frac{4\pi T}{\Phi_0 d}\ln(Tg_{el}), \tag{33}$$

where $H_{c1}^0(T) = 4\pi e_0/\Phi_0 = (\Phi_0/4\pi\lambda^2)\ln\kappa$ is the usual expression for H_{c1}. It can be seen that the fluctuation correction to $H_{c1}^0(T)$ is practically linear in T and is similar to the correction obtained for three-dimensional superconductors or for layered superconductors with the Josephson coupling of the VPs [2, 4, 5].

To obtain a correction to formula (33) in the case of small U_p/T (high temperatures), we extract the step-like function $g_{el}(E)$ from the density of states $g(E)$ given by Eq. (30), $g(E) = g_{el}(E) + \Delta g(E)$. The function $\Delta g(E)$ thus obtained coincides with $g(E)$ at $E < 0$, is antisymmetric in E, and differs from zero in a region of the order of U_p (see Fig. 2). Then the partition function $Z(E)$ in Eq. (26) can be written as

$$Z(E) \approx g_{el}T\exp\left(\frac{E}{T}\right)\left(1-\frac{1}{T^2}\int\limits_E^{|E|}E'dE'\frac{\Delta g(E')}{g_{el}} + \frac{1}{T}\int\limits_{|E|}^{\infty}dE'\frac{\Delta g(E')}{g_{el}}\right), \tag{34}$$

where $E < 0$; $\exp(-E'/T)$ is here replaced with $1-(E'/T)$, and we keep only the largest nonzero terms. Inserting this expression in Eq. (28) gives

$$\ln Z \approx \frac{E_{pin}}{T}+\ln(g_{el}T)-\frac{U_p^2}{T^2}\int\limits_{-\infty}^{E_0}g_m(E)I_2\left(\frac{E}{U_p}\right)dE - \frac{U_p}{T}\int\limits_{-\infty}^{E_0}g_m(E)I_1\left(\frac{E}{U_p}\right)dE, \tag{35}$$

where

$$I_1(x) = \int\limits_{-\infty}^{x}dt\frac{1+\mathrm{erf}(t)}{2} = \frac{x[1+\mathrm{erf}(x)]}{2} + \frac{1}{2\sqrt{\pi}}e^{-x^2}, \tag{36}$$

$$I_2(x) = \int\limits_x^0 t[1+\mathrm{erf}(t)]dt = -\frac{x^2[1+\mathrm{erf}(x)]}{2} - \frac{x}{2\sqrt{\pi}}e^{-x^2} + \frac{\mathrm{erf}(x)}{4}. \tag{37}$$

The first term in Eq. (35) cancels the term E_{pin}/d in formula (27). The second term in Eq. (35) leads to the thermal-fluctuation correction to H_{c1}, Eq. (33). In the third term in Eq. (35), we have $I_2(x) \approx -1/4$ at large δ, and this term is approximately equal to $U_p^2/4T^2$. As regards the last term in Eq. (35), it is small and can be neglected compared to the third term in the region $U_p < T < U_p\delta/4$. Indeed, we have $I_1(x) \le I_1(x_0) = E_{em}q/U_p$ (see Eq. (15)). Hence, the correction to $\ln Z$ associated with this term is of the order of $E_{em}q/T$. Eventually, we arrive at the following pinning correction to H_{c1}^T:

$$H_{c1} - H_{c1}^T \approx -\frac{\pi U_p^2}{\Phi_0 Td}, \tag{38}$$

which is quadratic in U_p and decreases with increasing the temperature ($U_p \propto \xi^{-1}\lambda^{-2}$, see the Appendix in Ref. [21]).

If the temperature is so low that $U_p/T \gg 1$, the second term in Eq. (27) is larger than the third one, and the lower critical field H_{c1} is mainly renormalized by pinning,

$$H_{c1} - H_{c1}^0 \approx \frac{4\pi E_{pin}}{\Phi_0 d} = \frac{\pi U_p}{\Phi_0 d} A\,, \qquad (39)$$

where the dimensionless factor $A \equiv 4E_{pin}/U_p$,

$$A = \delta \left[(x_0^2 - 0.5)[1 + \mathrm{erf}(x_0)] + \frac{x_0}{\sqrt{\pi}} \exp(-x_0^2) \right], \qquad (40)$$

weakly depends on δ (see Fig. 1). Hence, at low temperatures, the pinning correction to H_{c1} is practically proportional to the pinning strength U_p.

We emphasize that the obtained effect of pinning on H_{c1} is substantially due to the absence of position correlations between the VPs in a vortex line of the layered superconductors, Eq. (6), and results from the specific form of $\varepsilon(k_z)$ in the case of the electromagnetic coupling of the VPs, Eq. (3). We note that even weak pinning ($\delta \lesssim 1$) would have an effect on H_{c1} in such layered superconductors. Indeed, because $E_{pin} \sim U_p^2/(E_{em} q)$ in the case of weak pinning (see Eq. (24) and Fig. 1), we obtain from Eq. (39) at low temperatures that $H_{c1} - H_{c1}^0$ is quadratic in U_p. Thus, the difference in the renormalization of H_{c1} in the cases of weak and strong pinning is only in the magnitude of the effect.

4.3. Temperature dependence of H_{c1}

We next consider the temperature dependence of H_{c1}, Fig. 3. This dependence has been calculated numerically with both pinning and thermal fluctuations of the VPs taken into account,

$$H_{c1}(T) = H_{c1}^0(T) + \frac{4\pi E_{pin}}{\Phi_0 d} - \frac{4\pi T}{\Phi_0 d} \ln Z\,. \qquad (41)$$

In constructing Fig. 3, the following temperature dependences of λ and U_p were assumed: $\lambda(T)/\lambda(0) = (1 - t^2)^{-1/2}$, $U_p \propto \xi^{-1}\lambda^{-2}$ [21], and $\kappa \equiv \lambda(T)/\xi(T) = 70$, where $t = T/T_c$ with $T_c = 90$ K. For comparison, this figure also shows the lower critical field $H_{c1}^T(T)$ renormalized by thermal fluctuations only, Eq. (33), and $H_{c1}(T)$ calculated within a simplified approach. In that approach, averaging over the layers in Eqs. (19) and (28) is replaced by the formulas $E_{pin} = E_0$ and $\ln Z = \ln Z(E_0)$. In other words, it is assumed that at $T = 0$, the VPs in different layers are all in the same state with the energy $E = E_0$. It can be seen that the simplified approach does not disturb H_{c1} essentially, and hence this simplification can be successfully used in calculations of $H_{c1}(T)$ at $\delta \gg 1$.

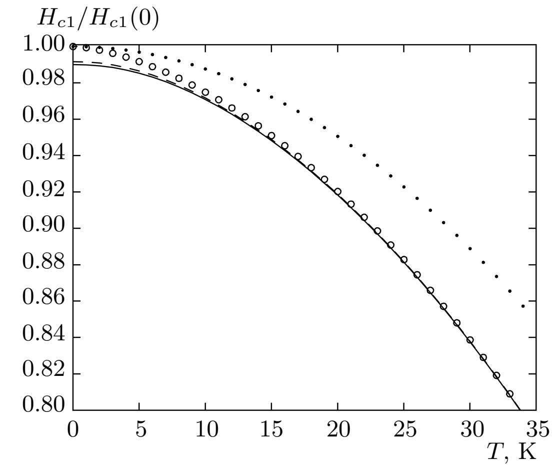

Fig. 3. The dependence $H_{c1}(T)$ calculated with Eq. (41) (solid line) in the case of the purely electromagnetic coupling of the VPs, i. e., for $T < T_J$. The dashed line shows $H_{c1}(T)$ within the simplified approach, the dotted line is $H_{c1}^0(T) = 4\pi e_0/\Phi_0$, and the circles give $H_{c1}^T(T)$, Eq. (33). Here, $\varepsilon = 1/500$, $d = 1.5$ nm, $U_p(0) = 20$ K, $\kappa = 70$, $\lambda(0) = 0.2\mu$m, $s_0 = \pi\xi^2$, and the temperature dependences of λ and U_p are presented in the text. These values of the parameters give $H_{c1}(0) \approx 169$ G, $E_{em}(0) \approx 0.14$ K, $q(0) \approx 1.77$, $\delta(0) \approx 80$, $p(0) \approx 0.84$, $T_J \approx 50$ K, and $T_{dp} \approx 25$ K

We note that if $p \equiv \pi\varepsilon\lambda/d < 1$ at $T = 0$, then a crossover temperature $T_J < T_c$ necessarily exists at which the parameter $p(T)$ reaches unity, $p(T_J) = 1$. This is due to the divergence of $\lambda(T)$ as $T \to T_c$. When $\lambda(T) \propto [1 - (T/T_c)^2]^{-1/2}$, we find

$$T_J = T_c\sqrt{1 - p(0)^2}\,. \qquad (42)$$

The results in this section are valid at $T < T_J$ (H_{c1} at $T > T_J$ is considered in Sec. 5.1). For the parameters in Fig. 3, we have $T_J \approx 50$ K, and the data of this figure show that the effect of pinning on H_{c1} dies out completely at temperatures lower than T_J. To identify the characteristic temperature at which the pinning becomes negligible, we define the so-called depinning temperature T_{dp} [14] by the equation

$$T_{dp} = |E_{pin}(T_{dp})|, \qquad (43)$$

where E_{pin} is given by Eq. (19). At temperatures higher than this T_{dp}, the VPs easily jump out of their pinning wells, the VP pinning becomes ineffective, and we can neglect this pinning in analyzing H_{c1}.

5. EFFECT OF JOSEPHSON COUPLING OF THE VORTEX PANCAKES ON H_{c1}

5.1. $T_J > T_{dp}$

Assuming that $T_J > T_{dp}$, we consider the temperature dependence of H_{c1} at $T > T_J$. In this temperature range, we have $p > 1$, and besides, the VP pinning is negligible, i. e., $H_{c1} = H_{c1}^T$. When $p > 1$, the vibrating modes with $k_z < k_\lambda$ lead to an uncorrelated motion of vortex segments of the length $L_\lambda = (k_z^{max}/k_\lambda)\, d = \pi\varepsilon\lambda$ (see Sec. 2), and the contribution of these longwave modes to the free energy f is

$$f_1 = -\frac{T}{L_\lambda}\ln\left(\frac{Tg_{el}d}{L_\lambda}\right). \tag{44}$$

This expression generalizes formula (32). On the other hand, the Josephson coupling of the VPs comprising a vortex line occurs for vibration modes with $k_z^{max} > k_z > k_\lambda$. These modes generate an internal motion of the vortex segments, and they give the following contribution to f:

$$f_2 = -\frac{T}{\pi}\int\limits_{k_\lambda}^{\pi/d} dk_z \ln\left(\frac{T\pi}{\varepsilon_l d s_0 k_z^2}\right) = -\frac{T}{d}\ln\left(\frac{Te^2 d}{\varepsilon_l s_0 \pi}\right) + \frac{T}{L_\lambda}\ln\left(\frac{Te^2\pi}{\varepsilon_l d s_0 k_\lambda^2}\right). \tag{45}$$

To display the difference between the total thermal part of the free energy, $f_1 + f_2$, and f_T given by Eq. (32), we represent $f_1 + f_2$ in the form $f_T + \Delta f_T$ where

$$\Delta f_T \equiv f_1 + f_2 - f_T = \frac{T}{d}R(p), \tag{46}$$

$$R(p) = 2\ln(p/e) + \frac{1}{p}\ln(e^2 p). \tag{47}$$

The function $R(p)$ is equal to zero at $p = 1$ and increases monotonically with increasing p for $p > 1$. Eventually, we obtain the following H_{c1}^T in the case of $p > 1$,

$$H_{c1}^T = H_{c1}^0(T) - \frac{4\pi T}{\Phi_0 d}\ln(Tg_{el}) + \frac{4\pi T}{\Phi_0 d}R(p). \tag{48}$$

Formulas (48) and (33) respectively describe H_{c1} at the temperatures $T > T_J$ and $T_J > T > T_{dp}$. At $T = T_J$, according to these formulas, a break in the temperature dependence $H_{c1}^T(T)$ occurs due to the term proportional to $R(p)$. The appropriate jump of dH_{c1}^T/dT at this point is equal to

$$\Delta\left[\frac{dH_{c1}^T}{dT}\right] = \frac{4\pi T_J}{\Phi_0 d}\left.\frac{d(\ln\lambda(T))}{dT}\right|_{T=T_J} \tag{49}$$

and is completely determined by the temperature dependence of λ in the vicinity of the point $t_J = T_J/T_c$,

$$\Delta\left[\frac{dH_{c1}^T}{dT}\right]\left[\frac{\text{mG}}{\text{K}}\right] = \frac{86.7}{d\,[\text{nm}]}\frac{t_J f'(t_J)}{f(t_J)}, \tag{50}$$

where $f(t) \equiv \lambda(t)/\lambda(0)$ and $f'(t) \equiv df/dt$. We note that this jump is relatively small,

$$\Delta\left[\frac{dH_{c1}^T}{dT}\right] = -\frac{2T_J}{e_0(T_J)\,d}\left.\left(\frac{dH_{c1}^0}{dT}\right)\right|_{T=T_J} \ll \ll \left|\frac{dH_{c1}^0}{dT}\right|_{T=T_J}. \tag{51}$$

For example, for the parameters in Fig. 3, we find that $2T_J/e_0(T_J)\,d \approx 0.049$, and $\Delta[dH_{c1}^T/dT] \approx 26$ mG/K.

Finally, we emphasize that we have obtained a sharp break in $H_{c1}^T(T)$ at the crossover temperature T_J because our model dependence $\varepsilon_l(k_z)$ described by Eqs. (2) and (3) also exhibits a break. It is clear that the break in $H_{c1}^T(T)$ can be somewhat smoothed in the case of the more realistic dependence (1) for $\varepsilon_l(k_z)$. Indeed, using this dependence (1), we can find the thermal part of the free energy and the appropriate $R(p)$:

$$R(p) = \ln\left(\frac{1+p^2}{e^2}\right) + \frac{2\,\mathrm{arctg}(p)}{p},$$

which is now defined at $p < 1$ as well. In the vicinity of the point $p = p_c = 0.3$, this $R(p)$ is close to the function $R(p < p_c) = 0$, $R(p \ge p_c) = 0.4(p - p_c)$, which has a break at a renormalized T_J defined by $p(T_J) = p_c$. Below, we disregard the effects associated with the smoothing of the break and with the renormalization of T_J, and, for simplicity, use Eqs. (2) and (3) only.

5.2. $T_J < T_{dp}$

We next consider H_{c1} for the opposite relation between the temperatures T_{dp} and T_J. When $0 < T_J < T_{dp}$, the parameter p exceeds unity in the temperature range where pinning is not negligible in general. At $T < T_J$, the formulas in Sec. 4 are valid for the calculation of H_{c1}. At $T > T_J$, the characteristic elastic energy of a VP is $E_{em}qp^2$ rather than $E_{em}q$. Because pinning of the VPs is implied to be strong compared with this elastic energy, we have $L_c = d$ for the Larkin length L_c. Thus, for temperatures $T_J < T < T_{dp}$, the VPs comprising the vortex line predominantly sit in the wells of the pinning potential, their positions are not correlated due to strong pinning, and $H_{c1}(T)$ is still given by the formulas in Sec. 4 if we replace q with qp^2. At $T = T_J$, the temperature dependence of H_{c1} has a break similar to that in the case $T_J > T_{dp}$ considered above. In particular, if $0 < T_J \ll T_{dp}$, the appropriate jump in dH_{c1}/dT can be estimated using formulas (15), (20), and (39):

$$\Delta \left[\frac{dH_{c1}^p}{dT} \right] \approx \frac{8\pi(E_0 - E_{pin})}{\Phi_0 d} \left. \frac{d(\ln \lambda(T))}{dT} \right|_{T=T_J}. \quad (52)$$

Similarly to the case $T_J > T_{dp}$, this jump is relatively small.

We now consider H_{c1} in the vicinity of the depinning temperature T_{dp}, assuming that $T_J < T_{dp}$. In the vicinity of T_{dp}, the Larkin length sharply increases [14]. When it reaches L_λ, a further increase in L_c does not occur because vortex deformations are uncorrelated on the scales larger than L_λ. This means that at $T \sim T_{dp}$, a crossover from strong pinning of individual VPs to the regime of pinning of vortex segments of the length L_λ occurs. At this crossover, the change $\Delta f_T = f_1 + f_2 - f_T(p > 1)$ in the thermal part of the free energy can be estimated as

$$\Delta f_T \approx \frac{T}{d} \left(-2 + \frac{\ln(e^2 p)}{p} \right), \quad (53)$$

where we have taken into account that at $T_J < T < T_{dp}$, the thermal contribution to the free energy has the form

$$f_T(p > 1) = -\frac{T}{d} \ln \left(\frac{T g_{el}}{p^2} \right) \quad (54)$$

due to the replacement of q with qp^2 in Eq. (29). At the crossover, this change Δf_T is accompanied by a positive change in the pinning energy Δf_{pin}. Indeed, above T_{dp}, most of the VPs in the vortex line easily leave the pinning wells, and the effect of the pinning energy on H_{c1} decreases. An interplay of this positive change in the pinning energy Δf_{pin} with the negative Δf_T given by Eq. (53) produces a "step" $\Delta H_{c1} \approx 4\pi(\Delta f_T + \Delta f_{pin})/\Phi_0$ in the temperature dependence of H_{c1} at $T \approx T_{dp}$ in addition to the difference in dH_{c1}/dT for points above and below T_{dp}. Of course, in reality, this step is smeared, and its temperature width can be roughly estimated as $E_0 - E_{pin}$. Moreover, for the smeared step, the interplay of the thermal and pinning contributions to the free energy can in principle result in an internal structure of this step.

To illustrate the behavior of H_{c1} near the depinning temperature, the dependence $H_{c1}(T)$ at $T < T_{dp}$ has been calculated numerically using formulas in Sec. 4

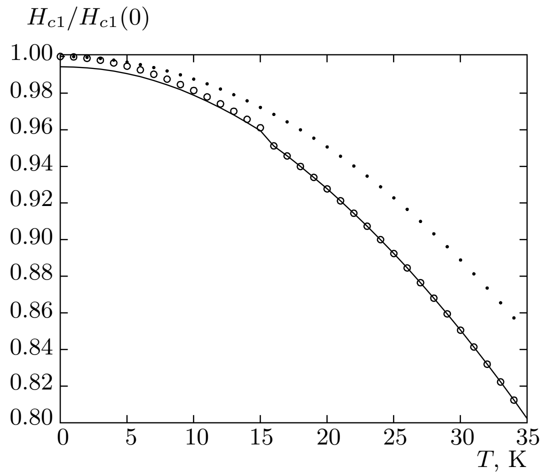

Fig. 4. Dependence $H_{c1}(T)$ in the case $p > 1$ (solid line). The circles depict $H_{c1}^T(T)$ calculated according to Eq. (48) at $T \geq T_{dp}$ and to $H_{c1}^T(T) = H_{c1}^0(T) + 4\pi f_T(p > 1)/\Phi_0$ at $T < T_{dp}$, where $f_T(p > 1)$ is given by Eq. (54). The dotted line is $H_{c1}^0(T) = 4\pi e_0/\Phi_0$. The parameters are the same as in Fig. 3, but $\varepsilon = 1/150$. This leads to $p(T = 0) \approx 2.79$ (i.e., $T_J = 0$) and $T_{dp} \approx 15$ K, $E_0 - E_{pin} \approx 6$ K. The smearing of the sharp jump in H_{c1} is only due to the temperature grid used in the calculations here

with the replacement $q \to qp^2$ for a temperature at which $p(T) > 1$. On the other hand, at $T > T_{dp}$, we neglect the pinning completely, and $H_{c1}(T)$ has been estimated with formula (48). The obtained results for two values of ε are presented in Figs. 4 and 5. In the case of Fig. 4, we have $T_J = 0$, i.e., $p(T) > 1$ at all temperatures. The value $p(T_{dp}) \approx 2.83$ noticeably exceeds unity, and at $T = T_{dp}$, the quantity Δf_T defined by Eq. (53) reaches a relatively large negative value $\Delta f_T \sim -T_{dp}/d$ that exceeds the positive Δf_{pin}. This leads to the negative step in $H_{c1}(T)$ that is visible in Fig. 4 at $T \approx 15$ K. On the other hand, in the case of Fig. 5, the temperature T_J lies in the interval from zero to T_{dp}; we have $p(T_{dp}) \approx 1.026$ and $\Delta f_T(T = T_{dp}) \approx -[p(T_{dp}) - 1]T_{dp}/d$, i.e., the absolute value of Δf_T at the point $T = T_{dp}$ is much less than the appropriate value in the case of Fig. 4, whereas Δf_{pin} does not change essentially. This leads to a positive step in $H_{c1}(T)$. We note that in both cases, the derivative dH_{c1}/dT for points above T_{dp} is less than for points below T_{dp}, i.e., $\Delta[dH_{c1}/dT] < 0$. However, this result is valid only for the points outside the crossover region. Inside the crossover region, the derivative dH_{c1}/dT can noticeably increase in the vicinity of T_{dp} (see the single point at $T = 25$ in the inset in Fig. 5).

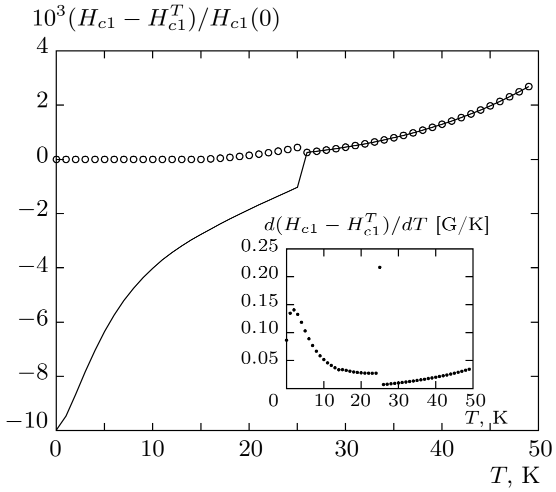

Fig. 5. The same dependences as in Fig. 4 but with $\varepsilon = 1/425$. This ε leads to $p(T = 0) \approx 0.986$, $T_J \approx 15$ K, $T_{dp} \approx 25$ K, and $E_0 - E_{pin} \approx 4$ K. For clarity, the dependences are shown as differences between the appropriate $H_{c1}(T)$ and the smooth function $H_{c1}^T(T)$ given by Eq. (33). The notation for the dependences is the same as in Fig. 4. In particular, the circles mark $H_{c1}^T(T) = H_{c1}^0(T) + 4\pi f_T(p > 1)/\Phi_0$ for $T_{dp} > T > T_J$ and depict dependence (48) at $T \geq T_{dp}$. For $T < T_J$, the circles correspond to Eq. (33). The inset: the derivative of the function shown by the solid line in the main panel. The tiny jump at $T = 15$ K is due to the break of $H_{c1}(T)$ at T_J

In principle, at $p(0) > 1$, one more specific situation can occur where, at some temperature $T_{cr} < T_{dp}$, the pinning energy decreases to the elastic energy $E_{em}qp^2 = q\Phi_0^2\varepsilon^2/16d\kappa^2$ that is practically independent of T. At this crossover temperature T_{cr}, the regime of strong pinning transforms into the regime of weak pinning, and the function $H_{c1}(T)$ should have a break. However, an analysis shows that for this situation to occur, the temperatures T_{cr} and T_{dp} have to be below $E_{em}qp^2$. For $d = 1.5$ nm, $\kappa = 70$, $q \sim 2$, and $\varepsilon \leq 1/100$, the elastic energy $E_{em}qp^2$ does not exceed 5 K. Since the pinning energy is practically independent of T at such low temperature, this situation is not realized for highly anisotropic superconductors with strong pinning, and we do not consider it here.

6. CONCLUSIONS

In this paper, we consider the lower critical field H_{c1} of layered superconductors with the purely electromagnetic ($p < 1$) or the electromagnetic and Josephson ($p > 1$) coupling of the vortex pancakes in a vortex line, with the parameter p defined in Eq. (4). It is found that vortex pinning by point defects leads to an additional renormalization of H_{c1} compared to the renormalization caused by thermal fluctuations of vortex pancakes. This effect of pinning is largely determined by the specific dependence of the vortex elasticity on the wave vector k_z for layered superconductors, Eqs. (1)–(3).

With the obtained results for H_{c1}, we analyze the temperature dependences of H_{c1} for various relations between the depinning temperature T_{dp} and the temperature T_J that marks the point at which the parameter p reaches unity. It is found that at $T = T_J$, the temperature dependence of H_{c1} exhibits a break. Besides, if $T_J < T_{dp}$, a break in $H_{c1}(T)$ may be accompanied in the vicinity of the depinning temperature by a smeared "step" in the temperature dependence of the lower critical field.

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