

PILGRIM DARK ENERGY IN $f(T)$ GRAVITY

M. Sharif*, S. Rani**

Department of Mathematics, University of the Punjab Lahore-54590, Pakistan

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We discuss the interacting $f(T)$ gravity with pressureless matter in an FRW spacetime. We construct an $f(T)$ model by following the correspondence scheme incorporating a recently developed pilgrim dark energy model and taking the Hubble horizon as the IR cutoff. We use constructed model to discuss the evolution trajectories of the equation-of-state parameter, the $\omega_T - \omega'_T$ phase plane, and state-finder parameters in the evolving universe. It is found that the equation-of-state parameter gives a phantom era of the accelerated universe for some particular range of the pilgrim parameter. The $\omega_T - \omega'_T$ plane represents freezing regions only for an interacting framework, while the ΛCDM limit is attained in the state-finder plane. We also investigate the first and second laws of thermodynamics assuming equal temperatures at and inside the horizon in this scenario. Due to the violation of the first law of thermodynamics in $f(T)$ gravity, we explore the behavior of the entropy production term. The validity of a generalized second law of thermodynamics depends on the present-day value of the Hubble parameter.

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1. INTRODUCTION

There is increasing evidence of dark energy (DE) over the last few years, which is assumed to be responsible for the accelerated expansion of the universe. This has been confirmed by a variety of observational constraints in the framework of different observational schemes [1]. The standard cosmology has been remarkably successful, but there remain some serious unresolved issues including the search for the best DE candidate. The origin and nature of DE is still unknown except in some particular ranges of the equation-of-state (EoS) parameter ω. In the absence of any solid argument in favor of a DE candidate, various approaches have been adopted such as dynamical DE models, and modified and higher-dimensional gravities.

The $f(T)$ theory of gravity [2] (the generalized teleparallel gravity, with T being the torsion scalar) attracted many people to explore it in different cosmological scenarios. This theory deals with torsion via the Weitzenböck connection (having zero curvature) instead of the Levi-Civita connection in general relativity, which is responsible for curvature. The $f(T)$ gravity has been studied extensively in application to many phenomena, e. g., the accelerated expansion of the universe [3], the correspondence (via quintessence, tachyon, K-essence, and dilaton scalar fields) carried out to discuss the dynamics of scalar fields as well as scalar potentials [4, 5] and to distinguish the $f(T)$ model from the ΛCDM model, state-finder diagnostics in a specific $f(T)$ model [6], validity/violation of the first and second laws of thermodynamics using the Wald entropy, corrected-entropy versions and magnetic field scenarios [7–9], and many more.

The search for a viable DE model is the basic key leading to the reconstruction phenomenon, particularly in modified theories of gravity. The corresponding energy densities are compared to construct the modified function in the underlying gravity. In this manner, the family of holographic reconstruction of DE models attains a significant place in discussing the accelerated expansion of the universe. Different $f(T)$ models were reconstructed via holographic DE (HDE) and new agegraphic DE (original and entropy corrected) models in [10]. The authors concluded that the corresponding EoS parameter gives consistent results in entropy-corrected models. In [11], an $f(T)$ model corresponding to the HDE model was obtained in a slightly different way. The authors found that the reconstructed model gives the phantom behavior as well as a unification of

*E-mail: msharif.math@pu.edu.pk
**E-mail: shamailatoor.math@yahoo.com

DE and dark matter. In [12], the reconstruction scheme was extended to a general (m,n)-type HDE in $f(T)$ as well as $f(R)$ gravity. The viability and cosmography of the obtained models were also discussed there.

Holographic DE has been attributed to the formation of black holes. Recent observations regarding the accelerating expansion of the universe are in favor of a phantom-dominated universe with no expectation of black holes. The idea of pilgrim DE (PDE) having the key point of a phantom-like universe to prevent the black hole formation was proposed in [13]. Recently, the behavior of interacting PDE models corresponding to the Hubble, event, and conformal age of the universe via different cosmological parameters such as EoS, $\omega-\omega'$ and state-finders was analyzed in [14]. The authors found consistent results for positive and negative values of the PDE parameter for these parameters.

In this paper, we construct the pilgrim $f(T)$ model via the reconstruction scheme and explore the EoS parameter, the $\omega_T-\omega'_T$ phase plane, and state-finder parameters. We also investigate thermodynamic laws for this model in $f(T)$ gravity for same temperature of the universe. This paper is arranged as follows. In Sec. 2, we briefly describe $f(T)$ gravity and its field equations, and construct a pilgrim $f(T)$ model. Section 3 is devoted to examining the evolution trajectories of some cosmological parameters. The validity of first and second laws of thermodynamics is investigated in this scenario in Sec. 4. In the last section, we summarize the results.

2. $f(T)$ GRAVITY AND PILGRIM DE MODEL

In this section, we first briefly discuss $f(T)$ gravity and its field equations, and then construct the pilgrim $f(T)$ model via the correspondence scheme.

2.1. The field equations

The action for $f(T)$ gravity [2] is defined as

$$\mathcal{S}=\frac{m_p^2}{2}\int d^4xh(f(T)+\mathcal{L}_m), \quad (1)$$

where $m_p^2=(8\pi\mathcal{G})^{-1}$ is the reduced Planck mass with $\mathcal{G}$ being the gravitational constant,

$$h=\sqrt{-g}=\det(h^a_\mu),$$

where g is the determinant of metric coefficients, h^a_μ is the tetrad field, and $\mathcal{L}_m$ is the Lagrangian density of matter in the universe. The tetrad field h^a_μ is related to the metric tensor as $g_{\mu\nu}=\eta_{ab}h^a_\mu h^b_\nu$, where $\eta_{ab}=\mathrm{diag}(1,-1,-1,-1)$ is the Minkowski space metrix, the indices (a,b) represent tangent space coordinates, (μ,ν) are the coordinate indices on the manifold, and all these indices range $0,1,2,3$. The variation of action (1) with respect to the tetrad yields the field equations

$$\left[h^{-1}\partial_\mu(hS_a{}^{\mu\nu})+h^\lambda_a T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu}\right]f_T+ \\ +S_a{}^{\mu\nu}\partial_\mu(T)f_{TT}+\frac{1}{4}h^\nu_a f=\frac{1}{2}\kappa^2 h^\rho_a\mathcal{T}^\nu_\rho, \quad (2)$$

where $f_T=df/dT$, $f_{TT}=d^2f/dT^2$, and $\mathcal{T}^\nu_\rho$ is the energy–momentum tensor of perfect fluid. The antisymmetric torsion and superpotential tensors are

$$T^\rho{}_{\mu\nu}=h^\rho_a(\partial_\nu h^a_\mu-\partial_\mu h^a_\nu),$$

$$S^{\mu\nu}{}_\rho=\frac{1}{4}[-T^{\mu\nu}{}_\rho+T^{\nu\mu}{}_\rho+T^{\mu\nu}{}_\rho+2\delta^\mu_\rho T^{\theta\nu}{}_\theta-2\delta^\nu_\rho T^{\theta\mu}{}_\theta],$$

which are used to define the torsion scalar as $T=T^\rho_{\mu\nu}S_\rho{}^{\mu\nu}$.

For a spatially flat Friedmann–Robertson–Walker (FRW) universe, a straightforward choice of the tetrad is

$$h^a_\mu=\mathrm{diag}(1,a(t),a(t),a(t)),$$

where $a(t)$ is a scale factor. This leads to the expression for the torsion scalar $T=-6H^2$, where $H=\dot{a}/a$ is the Hubble parameter and a dot represents the time derivative. The corresponding modified field equations are

$$12H^2f_T+f=2m_p^{-2}\rho, \quad (3)$$

$$48H^2\dot{H}f_{TT}-(12H^2+4\dot{H})f_T-f=2m_p^{-2}p. \quad (4)$$

Here, ρ and p denote the total energy density and pressure of the universe, satisfying the energy conservation equation

$$\dot{\rho}+3H(\rho+p)=0. \quad (5)$$

Equations (3) and (4) can be rewritten in terms of the usual Friedmann equations as

$$H^2=\frac{1}{3m_p^2}(\rho_m+\rho_T),\quad \dot{H}=\frac{1}{2m_p^2}(\rho_m+\rho_T+p_T), \quad (6)$$

where ρ_m is the matter contribution of energy density with pressureless matter ($p_m=0$), and torsion contributions ρ_T and p_T take the form

$$\rho_T=\frac{m_p^2}{2}(2Tf_T-f-T), \quad (7)$$

$$p_T = -\frac{m_p^2}{2}(-8\dot{H}Tf_{TT} + + (2T - 4\dot{H})f_T - f - T + 4\dot{H}). \quad (8)$$

In terms of fractional energy densities, the first equation in Eq. (6) can be expressed as

$$1 = \Omega_m + \Omega_T, \quad \Omega_m = \frac{\rho_m}{3m_p^2 H^2}, \quad \Omega_T = \frac{\rho_T}{3m_p^2 H^2}. \quad (9)$$

The nature and properties of DE and dark matter constitute one of the central problems in modern astrophysics. Dark energy as the most dominant component in the energy budget of the universe, having the possibility of nongravitational coupling to other components of the universe, in particular, to dark matter. This coupling results in modifying the background evolution of the dark sector, permitting any type of interaction to be constrained. There is no serious evidence presented up to now against this coupling. Here, we assume that pressureless matter (cold dark matter) interacts with the torsion component [15], and the corresponding nonconservation equations are given by

$$\dot{\rho}_m + 3H\rho_m = Q, \quad (10)$$

$$\dot{\rho}_T + 3H\rho_T(1 + \omega_T) = -Q, \quad (11)$$

where $\omega_T = p_T/\rho_T$ is the EoS parameter for the interacting $f(T)$ gravity and Q represents the interaction term that exchanges the energy between the torsion component and pressureless matter. In general, Q can be an arbitrary function of the energy densities of DE and pressureless matter as well as the Hubble parameter. Commonly, it induces simple choices of interaction [15, 16] such as

$$Q = 3dH\rho_m, \quad Q = 3dH\rho_{de}, \quad Q = 3dH(\rho_m + \rho_{de}),$$

where d is the coupling constant (interaction parameter). Some of these interactions are used for mathematical simplicity, while others have been proposed within some phenomenological approaches. The case $d = 0$ represents the noninteracting scenario. The sign of d is important in the sense that it reveals an exchange of energy: $d > 0$ implies that DE decays in dark matter, while $d < 0$ means that dark matter decomposes into DE. The positive coupling constant is favorable for the validity of thermodynamic laws. However, it was observed in [17] that Q must change its sign during the evolution of the universe from the deceleration to acceleration phase. Unfortunately, these choices for Q do not change their signs during the evolution, and this requires new interacting terms.

A new form of Q was introduced in [18] as

$$Q = 3dH(\rho_{de} - \rho_m). \quad (12)$$

As the universe evolves from the decelerated to accelerated regime, this interaction term changes its sign from negative to positive. Also, this form remains consistent from the thermodynamic standpoint. Using Eqs. (7), (9), and (12) in (11) and after some mathematical manipulations, we obtain the EoS parameter as

$$\omega_T = \frac{2Tf_T - f - T}{2Tf_T - f - T - \Omega_T T(2Tf_{TT} + f_T - 1)} \times \times \left[\frac{T(2Tf_{TT} + f_T - 1)}{2Tf_T - f - T} - 1 - d\left(1 - \frac{\Omega_m}{\Omega_T}\right)\right]. \quad (13)$$

In what follows, we adopt the reconstruction scenario to find a viable $f(T)$ model to discuss the evolution of this parameter.

2.2. Pilgrim $f(T)$ model

A well-known model was proposed in [19] as a possible candidate for DE with the help of an energy density bound named the HDE. To achieve the compatibility with an effective local quantum field, a relation between the ultraviolet (short-distance) and infrared (IR) (long-distance) cutoffs was given in [20] on the basis of limit set by the formation of a black hole. That is, for the quantum zero-point energy density ρ_Λ (which is the result of a short-distance cutoff), the total energy in a region of size L should not exceed the mass of a black hole of the same size, which requires the largest value of L to saturate this process. This is given by $\rho_\Lambda = 3c^2 m_p^2 L^{-2}$, where c is the holographic constant and L is the IR cutoff. Several choices of L have been proposed to distinguish different DE models within the holographic family such as Hubble, apparent, and event horizons, Granda–Oliveros cutoff [21], and so on.

Observations predict a phantom-like universe that undergoes a big-rip singularity (where all gravitationally bound objects are disrupted). On the other hand, the idea of an energy density bound came into being with the help of black hole formation in quantum gravity. It was found in many attempts that the black hole mass approaches zero or becomes zero when a phantom-like fluid accretes onto the black hole [22]. It would therefore be interesting to search for an appropriate phantom-like DE model that prevents the formation of black holes. This motivated Wei [13] to propose a phantom-like DE model called the pilgrim DE (PDE) model. This model inherits a strong repulsive force in order to prevent the formation of black holes. The PDE has the form

$$\rho_\Lambda = 3\epsilon^2 m_p^{4-u} L^{-u}, \qquad (14)$$

where ϵ and u are dimensionless constants. We use this model to obtain a viable $f(T)$ model under the reconstruction scheme. We note that different versions of the holographic family have been studied in $f(T)$ gravity to investigate more feasible results for the accelerating expansion of the universe.

We here assume Hubble horizon $L = 1/H$ as the IR cutoff to find an $f(T)$ model using Eq. (14). It was shown in [23] that the choice of the Hubble horizon as an IR cutoff in general relativity yields the same evolution of DE and dark matter (pressureless matter) and is therefore incompatible with the present status of the universe. However, with the passage of time, this deficiency has been resolved with the inclusion of an interacting scenario [24]. By imposing the correspondence of energy densities, $\rho_\Lambda = \rho_T$, we obtain the $f(T)$ model with

$$f(T) = T + \frac{c_1}{\sqrt{6}}(-T)^{1/2} + \\ + \frac{6^{1-u/2}\epsilon^2 m_p^{2-u}}{u-1}(-T)^{u/2}, \qquad (15)$$

where c_1 is an arbitrary constant, which can be determined in terms of boundary condition. It was argued in [25] that the gravitational constant $\mathcal{G}$ is replaced by an effective one in the nonlinear $f(T)$ gravity (in view of Eq. (13)). In this regard, the present-day value of $\mathcal{G}$ should be recovered from its effective value for a linear $f(T)$, which yields $f_T(T_0) = 1$. Here, $T_0 = -6H_0^2$ and H_0 is the present-day value of the Hubble parameter. Inserting the value of the first derivative of (15) in this condition, we found

$$c_1 = -\frac{6u\epsilon^2 m_p^{2-u}}{u-1} H_0^{u-1}. \qquad (16)$$

The holographic scenario becomes a physically viable model when the interaction between DE and dark matter is taken into account [26]. In this respect, the pilgrim $f(T)$ model may provide the viability in interaction with cold dark matter.

3. SOME COSMOLOGICAL PARAMETERS

Here, we examine the evolution of the EoS parameter, the behavior of $\omega_T - \omega'_T$, and the limit of the ΛCDM model using state-finder pair for the pilgrim $f(T)$ model in an interacting scenario.

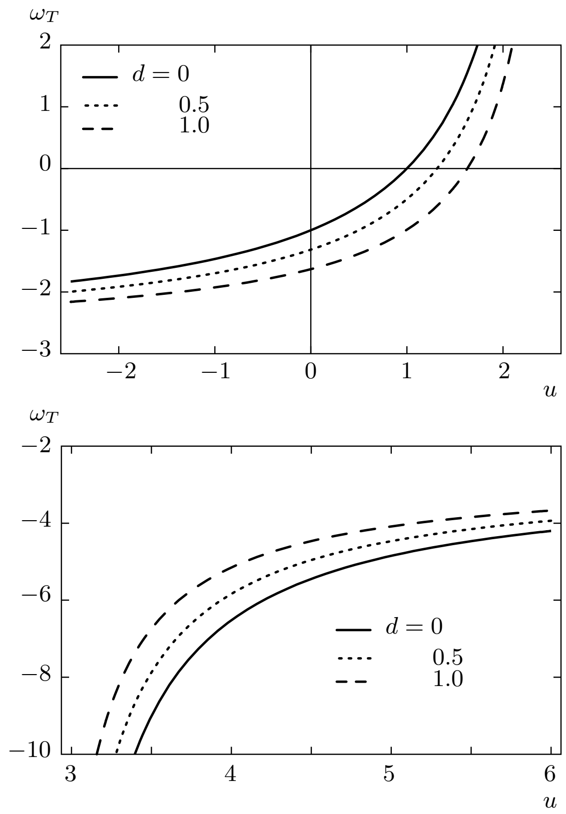

Fig. 1. Plot of the EoS parameter ω_T versus u for the pilgrim $f(T)$ model

3.1. The equation-of-state parameter

We explore the evolution of the EoS parameter for the pilgrim $f(T)$ model in interacting as well as noninteracting scenarios. Inserting Eq. (15) in (13), we have

$$\omega_T = \frac{2}{2 - u\Omega_T}\left(u - 1 - d\left(1 - \frac{\Omega_m}{\Omega_T}\right)\right). \qquad (17)$$

Its graph versus the PDE parameter u for different values of the interaction parameter d is shown in Fig. 1. We assume $d = 0, 0.5, 1$ and the present-day values of fractional energy densities $\Omega_m = 0.27$ and $\Omega_T = 0.73$. We plot ω_T versus two ranges of u similarly to the case in general relativity [14], i. e., $-2.5 \le u \le 2.5$ and $3 \le u \le 6$ in the upper and bottom panels in Fig. 1. Initially, the EoS parameter represents a phantom region of the universe for $u < 0$ for all values of d, as shown in the upper panel. As u increases, it crosses the phantom barrier $\omega_T = -1$ for $d = 0, 0.5, 1$ at the respective value $u = 0, 0.5, 1$. Thus, the EoS parameter for the pilgrim $f(T)$ model goes toward quintessence region and converges to the matter-dominated universe for $0 < u < 2.5$. On the other hand, in the bottom panel for $u > 3$, the evolution of the EoS parameter always stays in the phantom region of the universe. Thus, the accelerating expansion of the universe consistent with the observations is analyzed for the ranges $u < 0$ and $u > 3$ in interacting as well as noninteracting scenarios.

An $\omega_T - \omega'_T$ analysis

Here, we address the $\omega - \omega'$ phase plane (where ω' indicates the derivative of the EoS parameter with respect to $\ln a$) in order to elaborate the dynamical properties of the PDE model in $f(T)$ gravity. This phase plane was introduced in [27] in analyzing the evolving behavior of the quintessence DE model. The authors of [27] found that the area occupied by this DE model in the phase plane can be devided into thawing ($\omega' > 0$ when $\omega < 0$) and freezing regions ($\omega' < 0$ when $\omega < 0$). The $\omega - \omega'$ analysis has attracted many researchers for analyzing the dynamical behavior of different DE models such as quintom [28], phantom [29], quintessence [30], HDE [31], PDE [14] and so on.

Taking the derivative of Eq. (17) with respect to $\ln a$, we obtain

$$\omega'_T = \frac{6}{(2-u\Omega_T)^2} \times \times \left\{ u \left(u - 1 - d(1 - \frac{\Omega_m}{\Omega_T}) \right) [\Omega_T \omega_T (\Omega_T - 1) - - d(\Omega_T - \Omega_m)] + d(2 - u\Omega_T) \times \times \left[d\left(1 - \frac{\Omega_m^2}{\Omega_T^2}\right) + \omega_T \frac{\Omega_m}{\Omega_T} \right] \right\}. \quad (18)$$

The plot of ω'_T with respect to ω_T for the pilgrim $f(T)$ model is shown in Fig. 2, which indicates that this model meets the ΛCDM model only in the interacting case $d = 0.5$. However, the present values are $\omega'_T = 0.1, -0.1$ with respect to the value $\omega_T = -1$ for $d = 0,\ 1$. It is also observed that the $\omega_T - \omega'_T$ plane represents the thawing (in the noninteracting and interacting cases) and freezing regions (in the interacting case only).

3.3. State-finder diagnostics

Many DE models have been proposed in order to explain the accelerating expansion of the universe. However a sensitive test is required, which can differentiate between these models. The Hubble and deceleration ($q = -1 - \dot{H}/H^2$) are geometrical parameters that provide the expansion history of the universe but cannot differentiate between DE models. For this purpose, two new parameters called state-finders were introduced in [32] as

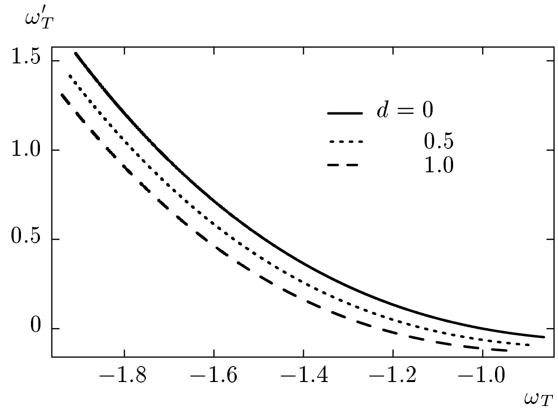

Fig. 2. Plot of $\omega_T - \omega'_T$ for the pilgrim $f(T)$ model

$$r = \frac{\dddot{a}}{aH^3}, \quad s = \frac{r-1}{3(q-1/2)}. \quad (19)$$

The parameter r can also be written in terms of q as $r = 2q^2 + q - q'$. These parameters exhibit well-known regions in the $s - r$ plane; for example, $(r, s) = (1, 0), (1, 1)$ show the ΛCDM and CDM limits, while the regions ($s > 0$ and $r < 1$) correspond to the phantom and quintessence DE. Also, these parameters distinguish different DE models from the ΛCDM model and the corresponding $r - s$ plane provides the distance of a given DE model from the ΛCDM limit.

The expressions for a state-finder pair for the pilgrim $f(T)$ model are

$$r = 1 - \frac{3}{2}\Omega_T\omega'_T + \frac{9}{2}\omega_T[\Omega_T(\omega_T + 1) + d(\Omega_T - \Omega_m)], \quad (20)$$

$$s = 1 + \omega_T - \frac{\omega'_T}{3\omega_T} + \frac{d}{\Omega_T}(\Omega_T - \Omega_m), \quad (21)$$

where ω_T and ω'_T are given in Eqs. (17) and (18). The plot of state-finder parameters is shown in Fig. 3 with the same assumptions for the cosmological parameters. This shows that the trajectories of $s - r$ in the noninteracting and interacting cases meet the ΛCDM limit. The trajectories also coincide with the behavior of the Chaplygin gas model (where $s < 0$ and $r > 1$). The quintessence and phantom DE regions are also obtained in this $s - r$ plane in the noninteracting and interacting cases. We note that the $s - r$ plane of the PDE model in $f(T)$ gravity is the combination of all possible existing well-known regions, which is an interesting feature.

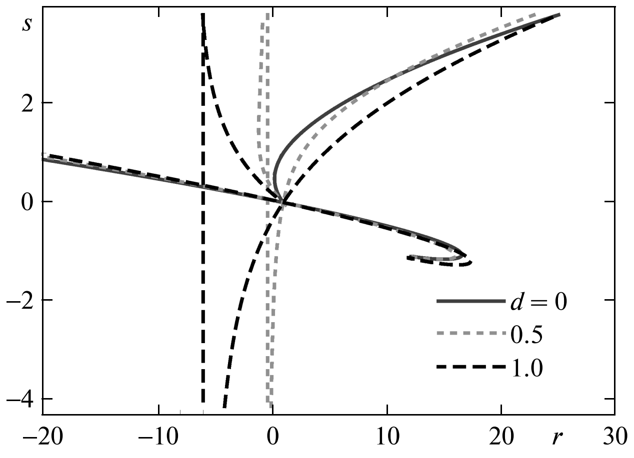

Fig. 3. Plot of state-finders for the pilgrim $f(T)$ model

4. THERMODYNAMICS

In this section, we discuss the nonequilibrium description [7, 33] of the first and second laws of thermodynamics in $f(T)$ gravity. Since this gravity is not locally Lorentz invariant, there alway exists extra degrees of freedom leading to the violation of the first law of thermodynamics [34, 35]. The system does not remain in equilibrium, which results in an entropy production term. To find this term, we rewrite Eqs. (3) and (4) for convenience as follows

$$\begin{aligned} H^2 &= \frac{1}{3m_p^2 f_T}(\rho_m + \bar{\rho}_T), \\ \dot{H} &= -\frac{1}{2m_p^2 f_T}(\rho_m + \bar{\rho}_T + \bar{p}_T), \end{aligned} \tag{22}$$

where $\bar{\rho}_T$ and $\bar{p}_T$ are the redefined torsion contributions

$$\begin{aligned} \bar{\rho}_T &= \frac{m_p^2}{2}(Tf_T - f), \\ \bar{p}_T &= \frac{m_p^2}{2}(4Hf_{TT} - Tf_T + f). \end{aligned} \tag{23}$$

The corresponding continuity equation yields

$$\dot{\bar{\rho}}_T + 3H(\bar{\rho}_T + \bar{p}_T) = -\frac{m_p^2 T}{2} f_{TT}, \tag{24}$$

which implies the nonconservation equation because $f_{TT} \neq 0$ in a nonequilibrium system. We assume here that the boundary of the universe is covered by the dynamical apparent horizon R_A [36] for which the Hawking temperature is given by

$$T_A = \frac{1}{2\pi R_A}\left(1 - \frac{\dot{R}_A}{2HR_A}\right).$$

For a flat FRW spacetime, it reduces to the Hubble horizon. Using Eq. (22) in the time derivative of the Hubble horizon, $\dot{R}_A = -H\dot{H}R_A$, yields

$$\frac{f_T}{4\pi\mathcal{G}}\frac{dR_A}{dt} = H(\rho_m + \bar{\rho}_T + \bar{p}_T)R_A^3. \tag{25}$$

In modified theories of gravity [37], the horizon entropy S is called the Wald entropy (related to the Noether charge method) and is expressed as $S = A/4\mathcal{G}_{eff}$. Here, $\mathcal{G}_{eff} = \mathcal{G}/f'$ is the effective gravitational coupling, f' is the derivative of f with respect to the corresponding argument, and $A = 4\pi R_A^2$ is the area of the horizon. The Wald entropy in $f(T)$ gravity is given by

$$S = \frac{Af_T}{4\mathcal{G}}, \tag{26}$$

which is also confirmed by a matter density perturbation through $\mathcal{G}_{eff}$. Taking time derivative of this equation and using (25), we obtain

$$\frac{1}{2\pi R_A}\frac{dS}{dt} = 4\pi H(\rho_m + \bar{\rho}_T + \bar{p}_T)R_A^3 + \frac{R_A}{2\mathcal{G}}\frac{df_T}{dt}. \tag{27}$$

Introducing the Hawking temperature in this equation yields

$$\begin{aligned} T_A dS = {} & 4\pi H(\rho_m + \bar{\rho}_T + \bar{p}_T)R_A^3 dt - \\ & - 2\pi(\rho_m + \bar{\rho}_T + \bar{p}_T)\, dR_A + \frac{\pi R_A^2}{\mathcal{G}} T_A df_T. \end{aligned} \tag{28}$$

The Misner–Sharp energy ($E = R_A/2\mathcal{G}$ in general relativity) can be modified accordingly as $E = R_A f_T/2\mathcal{G}$. For the Hubble horizon, it becomes

$$E = 3m_p^2 H^2 f_T V = (\rho_m + \bar{\rho}_T)V,$$

where $V = (4/3)\pi R_A^3$ is the volume inside the horizon. Its first derivative takes the form

$$\begin{aligned} dE = {} & -4\pi H(\rho_m + \bar{\rho}_T + \bar{p}_T)R_A^3 dt + \\ & + 4\pi(\rho_m + \bar{\rho}_T)R_A^2 dR_A + \frac{R_A}{2\mathcal{G}} df_T. \end{aligned} \tag{29}$$

By combining Eqs. (28) and (29), it follows that

$$\begin{aligned} T_A dS = {} & -dE + 2\pi R_A^2(\rho_m + \bar{\rho}_T - \bar{p}_T)\, dR_A + \\ & + \frac{R_A}{2\mathcal{G}}(1 + 2\pi R_A T_A)\, df_T. \end{aligned} \tag{30}$$

With the help of the energy–momentum tensor relation, the work density can be defined as [38]

$$\begin{aligned} W = {} & -\frac{1}{2}(\mathcal{T}_m^{\mu\nu} g_{\mu\nu} + \bar{\mathcal{T}}_T^{\mu\nu} g_{\mu\nu}) \Rightarrow W = \\ & = \frac{1}{2}(\rho_m + \bar{\rho}_T - \bar{p}_T). \end{aligned} \tag{31}$$

Using this expression in Eq. (30) and rearranging, we finally obtain

$$T_A dS + T_A dS_p = -dE + W dV, \qquad (32)$$

where the additional term dS_p is identified as the entropy production term in the nonequilibrium description of the thermodynamics in $f(T)$ gravity. It has the general form

$$dS_p = -\frac{R_A}{2T_A\mathcal{G}}(1 + 2\pi R_A T_A)\, df_T = \\ = \frac{6\pi}{\mathcal{G}}\frac{8HT + \dot{T}}{T(4HT + \dot{T})} df_T. \qquad (33)$$

It is obvious that this additional term vanishes for teleparallel gravity ($f(T) = T$), the same as in the case of general relativity.

To investigate the behavior of the entropy production term for the pilgrim $f(T)$ model, we take the second derivative of (15),

$$f_{TT} = \frac{u\epsilon^2 m_p^{2-u}}{24(u-1)}\left(\frac{H_0^2}{H^3} + \frac{u-2}{u-1}H^{u-4}\right).$$

Using this expression in Eq. (33), we have

$$\frac{dS_p}{dt} = \frac{6\pi}{\mathcal{G}}\frac{(8HT + \dot{T})\dot{T}}{T(4HT + \dot{T})} f_{TT} = \frac{\pi u\epsilon^2 m_p^{2-u}}{2\mathcal{G}(u-1)} \times \\ \times \frac{\dot{H}(4H^2 + \dot{H})}{2H^2 + \dot{H}}\left(\frac{H_0^2}{H^4} + \frac{u-2}{u-1}H^{u-5}\right). \qquad (34)$$

It is observed that for the universe expanding with acceleration, $H^2 \pm \dot{H} > 0$. Moreover, the phantom-like accelerating expansion of the universe corresponds to $\dot{H} > 0$, while for negative $\dot{H}$, quintessence-like behavior of the universe is obtained. The behavior of $\dot{S}_p$ in the evolving universe depends on the signs of $\dot{H}$ and u. We note that $u \neq 1$ in this case, whereas $u = 0$ gives a vanishing entropy production term that corresponds to the teleparallel gravity.

In the phantom-like accelerating universe, we observe that $\dot{S}_p > 0$ for $u < 0$ and $u > 2$, while the range $0 < u < 1$ represents the decreasing entropy-production term. For the range $1 < u < 2$, the behavior of the entropy production term depends on the strength of the involved terms, implying that $\dot{S}_p > 0$ if the first term dominates over the second term in Eq. (34), and $\dot{S}_p < 0$ otherwise. A similar but inverted behavior is observed for a quintessence-like accelerated expanding universe, i. e., $\dot{S}_p < 0$ for $u < 0$ and $2 < u$, and the range $0 < u < 1$ corresponds to $\dot{S}_p > 0$. The time derivative of the entropy production term becomes positive if the first term is dominated by the second one in the range $1 < u < 2$, and turns out to be negative otherwise. Thus, the pilgrim $f(T)$ model takes different ranges of the model parameter u for the validity of the first law of thermodynamics. The teleparallel gravity may be recovered with the passage of time for some ranges of u if $\dot{S}_p$ decreases and approaches zero. It is argued in [7] that with this type of behavior, entropy production is not a permanent phenomenon.

Finally, we investigate the validity of the generalized second law of thermodynamics in this scenario. If we examine the increasing behavior of the total entropy of the horizon (which includes the horizon entropy in addition to the entropy of total matter), it implies the validity of the generalized second law of thermodynamics. The Gibbs equation for the entropy of total matter inside the horizon is given by

$$T_A dS_{in} = d(V\bar{\rho}_T) + \bar{p}_T dV = V dt + (\bar{\rho}_T + \bar{p}_T)\, dV. \qquad (35)$$

Here, we assume the same temperature inside and outside the apparent horizon [7, 33]. Combining Eqs. (22), (32), and (35), we express the time derivative of the total entropy of the horizon as

$$\dot{S} + \dot{S}_p + \dot{S}_{in} = \frac{\dot{H}^2}{2\mathcal{G}H^4} f_T. \qquad (36)$$

For the pilgrim $f(T)$ model in this equation, the final expression becomes

$$\dot{S} + \dot{S}_p + \dot{S}_{in} = \frac{\dot{H}^2}{2\mathcal{G}H^4} + \\ + \frac{u\epsilon^2 m_p^{2-u}}{4\mathcal{G}(u-1)H^5}(H_0^{u-1} - H^{u-1}). \qquad (37)$$

This equation implies that the generalized second law of thermodynamics is satisfied for the present-day value of the Hubble parameter with $u > 0$, $u \neq 1$ regardless the sign of $\dot{H}$.

5. CONCLUDING REMARKS

We have studied the interacting $f(T)$ gravity with pressureless matter in an FRW universe using the reconstruction scheme to discuss the evolution of the universe. For this purpose, we have used a recently proposed PDE model having a strong repulsive force in order to take the universe to big-rip singularity without formation of black holes. The Hubble horizon is taken as the IR cutoff, which gives consistent results with interaction. To discuss the pilgrim $f(T)$ model, we have investigated the evolution trajectories of the EoS parameter, the $\omega_T - \omega_T'$ phase plane, and the state-finder parameters. For three values of the interaction parameter $d = 0, 0.5, 1$, the parameter ω_T versus the PDE parameter u represents consistent results for an accelerating universe (Fig. 1). In this regard, we have obtained two ranges of u, i. e., $u < 0$ and $u > 3$ for both interacting and noninteracting backgrounds. These ranges indicate a phantom-dominated universe, where no possibility exists for the formation of a black hole.

The evolution trajectory of the $\omega_T - \omega'_T$ plane incorporating the pilgrim $f(T)$ model represents the ΛCDM limit only for $d = 0.5$. It yields thawing regions for all values of the interaction parameter and freezing regions only in the interacting case. Also, the statefinder parameters in $s - r$ plane are found to meet all possible existing regions (quintessence, phantom, and Chaplygin gas). Finally, we have investigated the validity of the first and second laws of thermodynamics in the nonequilibrium background under the assumption of the same temperature of the universe. It is found that the first law of thermodynamics is violated in $f(T)$ gravity due to the lack of local Lorentz invariance, which results an entropy production term. We have analyzed the behavior of this entropy production term as well as the validity of the generalized second law of thermodynamics for the pilgrim $f(T)$ model. It is found that for a phantom-like universe ($\dot{H} > 0$), the entropy production term decreases for $0 < u < 1$ and increases for $u < 0$, $2 < u$. However, its behavior depends on the strength of the involved terms within the range $1 < u < 2$. For a quintessence-like universe, all the results are inverted for the same ranges of the PDE parameter u.

In general relativity, the PDE model provides phantom-like behavior with a Hubble horizon only with $d = 1$ for all values of the PDE parameter u [14]. For the pilgrim $f(T)$ model, the phantom-like universe is attained for $u < 0$ and $u > 3$ in the interacting as well as noninteracting cases. It is interesting to mention here that our results are consistent with those in [13, 14] for $u < 0$. In the $\omega - \omega'$ analysis, the PDE model meets the ΛCDM limit only in the noninteracting case, with a freezing region in the interacting case, while this limit and region are obtained only in the interacting case for the pilgrim $f(T)$ model. The $s - r$ plane recovers all the existing regions corresponding to fixed values of (s, r) for both models. We can check the cosmological evolution and thermodynamic behavior of the pilgrim $f(T)$ model by taking the event horizon as the boundary of the universe. The basic purpose in developing the pilgrim model is to explain the fate of black holes in the presence of a large amount of phantom energy in the universe. Thus, it would be an interesting and attractive idea to constrain model parameters via modified theories of gravity by the cosmographic technique. This helps in solving many cosmological issues, offering a glimpse of one of the notions (phantom energy) of the universe in the later times.

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