TETRADS IN SOLIDS: FROM ELASTICITY THEORY TO TOPOLOGICAL QUANTUM HALL SYSTEMS AND WEYL FERMIONS

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1. INTRODUCTION

Theory of elasticity in topological insulators has many common features with relativistic quantum fields interacting with gravitational field in the tetrad form. Here we discuss several issues in the effective topological (pseudo)electromagnetic response in three-dimensional weak crystalline topological insulators with no time-reversal symmetry that feature elasticity tetrads, including a mixed "axial-gravitational" anomaly. This response has some resemblance to "quasitopological" terms proposed for massless Weyl quasiparticles with separate, emergent fermion tetrads.

We demonstrate the principal difference between the elasticity tetrads and the Weyl fermion tetrads in the construction of the topological terms in the action. In particular, the topological action expressed in terms of the elasticity tetrads, cannot be expressed in terms of the Weyl fermion tetrads since in this case the gauge invariance is lost.

There are several sources of emergent tetrad gravity in solids. The tetrad field in particular emerges in Dirac and Weyl semimetals in the vicinity of nodes in the electron spectrum (the fermion tetrads or Weyl tetrads). A different set of tetrads (the elasticity tetrads) emerge in the theory of elasticity, see e. g. Refs. [1,2]. Our consideration of the tetrad fields is based on approach formulated in two books by Landau and Lifshitz [2, 3] from their multi-volume "Course of Theoretical Physics". Lev Petrovich Pitaevskii was not only one who had completed the course but later also the editor of the books in the series.

More specifically, we discuss the role of the crystalline tetrads, torsion (dislocations) and gauge invariance in the response of $3+1$-dimensional ($D = 3+1$) weak topological insulators with anomalous quantum Hall effect (AQHE). We contrast this to a similar term for the fermion tetrads in Weyl semimetals/superfluids with the chiral/axial anomaly. For the topological AQHE response, we obtain a $D = 2n+2 = 6$ mixed axial-gravitational anomaly, and consider its dimensional reduction to $D = 2n+1 = 5$ and $D = 2n = 4$ anomalous actions, as well as the extension to driven Floquet-Bloch systems, which are expressed in terms of three or four integer topological invariants in the crystal 4-momentum space.

We also note the possibility of an emergent fermion tetrad gravity that is different for left- and right-handed Weyl fermions. This is possible in condensed matter systems although usually precluded by discrete symmetries. In the high-energy particle physics vacuum, the effect is constrained by (discrete) Lorentz symmetries.

2. WEYL TRIADS AND TETRADS

In condensed matter, see e. g. reviews in Refs. [4,5], the quasirelativistic equivalent of a Weyl Hamiltonian can arise in 3+1d fermionic systems with topologically protected nodes in the spectrum — the monopoles in the Berry phase [6]. This can occur in topological su-

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perfluids, superconductors and semimetals [5,7,8]. Depending on the physical symmetries and the (emergent) gauge fields at the node, the discrete symmetry transformations may take different forms that for elementary particles. Moreover, it is possible to consider discrete crystalline symmetries as well, leading to even more possibilities. The Hamiltonian $H(\mathbf{p})$ describing emergent Weyl fermions near the node in momentum space $\mathbf{p}$ contains effective triad field e^k_a, effective or real gauge field A_k, and dimensionless charge q [8–11]:

$$H(\mathbf{p}) = e^k_a \sigma^a (p_k - qA_k)\,. \quad (1)$$

Here, the σ^a with $a = 1,2,3$ are the standard Pauli matrices labeling spin, pseudospin, particle-hole or band index depending on the context. The four dimensional tetrad field e^μ_α is relevant when the fermion Green's function, or the action is considered:

$$G^{-1}(p_\mu) = e^\mu_\alpha \sigma^\alpha (p_\mu - qA_\mu)\,, \quad (2)$$

where $p_\mu = (\omega, -\mathbf{p})$, $\alpha = 0,1,2,3$ and $\sigma^\alpha = (\mathbb{1}, \sigma^a)$ as above. We call the fermion tetrad e^μ_α the Weyl tetrad — in order to distinguish them from the elasticity tetrads discussed later.

In Eq. (2), e^0_α is dimensionless, whereas e^a_k has dimensions of velocity (or dimensionless in units where $c = 1$). If there is no special symmetry between the Weyl nodes in the spectrum, the tetrads for the left and right handed fermions can have in principle different tetrads. And of course, the left and right tetrads have opposite signs of determinant, $e \equiv \det e^\alpha_\mu$. The topological Lifshitz transition with the change of the sign of the determinant — the transition to antispacetime via the space time with the degenerate tetrads — is described in Ref. [12].

From the condensed matter point of view, the chiral fermions of the massless Standard Model with their tetrad (semiclassical) gravity are not necessarily the primary objects, and the question "What is more fundamental: gravity or the chiral Weyl fermions?" becomes appropriate [13].

Two scenarios are possible depending on what is primary:

(i) If gravity is primary, then there should be the unique tetrad field e^μ_α, which is the same for both chiralities. This means that the two $SU(2)_{L,R}$ Lorentz representations are complex conjugate, $iK^a_{L,R} = \pm\sigma^a_{L,R}/2$ with $a = 1,2,3$, and must have different signs for right and left handed fermions.

(ii) If the chiral left and right handed fermions are primary, they may have different tetrad fields, $e^\mu_{\alpha R} \neq e^\mu_{\alpha L}$. If gravity has a unique spacetime geometry, there is a constraint that the two tetrads should comprise the same metric field:

$$g^{\mu\nu} = e^\mu_{aR} e^\nu_{bR} \eta^{ab} = e^\mu_{aL} e^\nu_{bL} \eta^{ab}\,, \quad (3)$$

which is invariant under the local Lorentz transformations $e^\mu_{\alpha R,L} \to \Lambda^\beta_\alpha e^\mu_{\beta R,L}$. In order to form a massive Dirac fermion, the tetrads must be connected by the discrete P, T, C symmetries that transform the left and right handed fermions into each other. For example, under parity $e^\mu_{0R} = e^\mu_{0L}$, $e^\mu_{1R} = -e^\mu_{1L}$, etc.

In the case (i), the Lagrangians for right and left Weyl fermions are expressed in terms of the same tetrads. Taking for simplicity only the diagonal tetrad matrices $e^\mu_\alpha = (e^0_0, e^k_a)$ one has:

$$\mathcal{L}_R = \Psi^+_R (e^0_0 p_0 + e^k_a \sigma^a p_k) \Psi_R, \quad (4)$$

$$\mathcal{L}_L = \Psi^+_L (e^0_0 p_0 - e^k_a \sigma^a p_k) \Psi_L\,. \quad (5)$$

The massive Dirac fermions are obtained after the electroweak Higgs transition when, due to the broken electroweak symmetry, matrix elements M between the chiral right and left handed fermions appear. The Lagrangian for Dirac fermions is

$$\mathcal{L} = \Psi^+ \left(e^0_0 p_0 + \tau_3 e^k_a \sigma^a p_k + M\tau_1\right) \Psi = \\ = \bar\Psi (\gamma^\alpha e^\mu_\alpha p_\mu + M)\Psi\,, \quad (6)$$

where the Pauli matrices τ^a operate in the L, R-chirality space and the chiral Dirac matrices satisfy $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$ with

$$\bar\Psi = \Psi^+\gamma^0, \quad \gamma^0 = \tau_1, \quad \gamma^a = i\tau_2\sigma^a, \quad a = 1,2,3. \quad (7)$$

In the case (ii), the chiral fermions are fundamental, and they generate tetrad fields, which are different for the two chiralities:

$$\mathcal{L}'_R = \Psi^+_R (e^{0R}_0 p_0 + e^{kR}_a \sigma^a p_k) \Psi_R, \\ \mathcal{L}'_L = \Psi^+_L (e^{0L}_0 p_0 + e^{kL}_a \sigma^a p_k) \Psi_L\,, \quad (8)$$

where $e^\mu_{\alpha R} \neq e^\mu_{\alpha L}$. The parity symmetric Dirac Lagrangian in Eq. (6) is obtained if there is an underlying symmetry between the right and left fermions, which leads to $e^0_{0R} = e^0_{0L} \equiv e^0_0$ and $e^k_{aR} = -e^k_{aL} \equiv e^k_a$ for $a = 1,2,3$. Then one has

$$\mathcal{L}_{Dirac} = \Psi^+ \left(e^0_0 p_0 + \tau_3 e^k_a \sigma^a p_k + M\tau_1\right) \Psi = \\ = \bar\Psi (\gamma^a e^\mu_a p_\mu + M)\Psi\,, \quad (9)$$

i. e. the same Dirac equation as Eq. (6).

3. AQHE IN WEAK TOPOLOGICAL INSULATORS AND WEYL SEMIMETALS

Consider a band insulator with broken time-reversal invariance in three dimensions. There are two possibilities: a trivial insulator and a weak topological insulator with an anomalous Hall effect [14], protected by crystalline translational symmetries, see e. g. the recent review in Ref. [15]. That is, the electromagnetic response of the $D = 3+1$-dimensional insulator may contain the following topological term:

$$S_{4D}[A_\mu] = \frac{1}{4\pi^2} \sum_{a=1}^{3} N_a \int d^4x \, E_\mu{}^a \epsilon^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta \,. \quad (10)$$

Here the $E_\mu{}^a(x) = (E_0{}^a(x), E_i{}^a(x))$, with $i, a = 1, 2, 3$ play the role of tetrad field with dimensions of momentum $[L]^{-1}$. The spatial $\mathbf{E}^a(x) = E_i{}^a(x)$ are primitive vectors of the reciprocal Bravais lattice, which depend on spacetime coordinates under deformations of the crystal lattice. In the absence of dislocations, one has $\partial_\mu E_\nu{}^a - \partial_\nu E_\mu{}^a = 0$ which guarantees the gauge invariance of Eq. (10) [16].

The integer coefficients N_a — three momentum space invariants — are expressed in terms of integrals of the Green's functions [17]:

$$N_a = \frac{\epsilon_{ijk}}{8\pi^2} \int\limits_{-\infty, BZ}^{\infty} d\omega \, dS_a^i \, \mathrm{Tr}[(G\partial_\omega G^{-1})(G\partial_{k_i} G^{-1}) \times \times (G\partial_{k_j} G^{-1})] \,. \quad (11)$$

The momentum integral is over the 2D torus in the cross section $\mathbf{S}_a$ of the three-dimensional Brillouin zone (BZ). The integer invariants N_a are topological invariants of the system and in particular remain locally well-defined under smooth deformations of the lattice. Under sufficiently strong deformations or disorder one can have regions of different $N_a(x)$ with chiral edge modes. In that case, the global invariant, if any, is defined by the topological charge of the dominating cluster which percolates through the system [18].

For Weyl semimetals the corresponding action Eq. (10) with QAHE can be found e. g. in Ref. [19] (Eq. (1)) and for relativistic Weyl fermions in Ref. [20]. However, in semimetals such an action is not appropriate in general due to the violation of gauge invariance. In semimetals, the separation q_μ between the Weyl nodes can depend on space and time under deformations, and one obtains:

$$S_{4D} = \frac{1}{4\pi^2} \int d^4x \; q_\mu(x) \epsilon^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta \,. \quad (12)$$

The Eq. (12) violates gauge invariance, unless q_μ is a constant of the topological medium/vacuum or there is an integrability constraint $\partial_\nu q_\mu - \partial_\mu q_\nu = 0$ [16].

Dislocations and torsion

The tetrad integrability constraint $\partial_\mu E_\nu{}^a - \partial_\nu E_\mu{}^a = 0$ is violated in the presence of topological defects of the crystal lattice such as dislocations, stacking faults and twin boundaries. In the presence of topological defects corresponding to dislocations, the density of dislocations plays the role of torsion:

$$\begin{aligned} T^a_{kl} &= (\partial_k E_l{}^a - \partial_l E_k{}^a), \\ T^m_{kl} &= E^m_a (\partial_k E_l{}^a - \partial_l E_k{}^a) \,. \end{aligned} \quad (13)$$

Note that this differs from the differential geometric torsion by a factor corresponding to the primitive lattice vectors $E_i^{(0)a}$. For a single dislocation one obtains, see e. g. Ref. [21] and references therein,

$$\partial_k E_l{}^a - \partial_l E_k{}^a = n^a \epsilon_{klp} \int ds \, \kappa^p \delta(\mathbf{r} - \mathbf{r}(s,t)) \,. \quad (14)$$

Here the n^a are integer topological charges of the dislocation, which enter the Burgers vector of the dislocation, $b^m = \sum_a E_a{}^m n^a$, and κ^p is the unit tangent vector along the corresponding dislocation line $\boldsymbol{\kappa} = d\mathbf{r}/ds$. The torsion is expressed in terms of the Burgers vector of the dislocation as

$$T^m_{kl} = b^m \epsilon_{klp} \int ds \, \kappa^p \delta(\mathbf{r} - \mathbf{r}(s,t)) \,. \quad (15)$$

The fermion zero modes on these topological defects provide the Callan–Harvey mechanism of anomaly cancellation [22–25]. The number of the 1D fermion zero modes on dislocation is the sum of the products of topological charges of the insulator N_a and topological charges (winding numbers) n^a of the dislocation [26]:

$$\nu_{ZM} = \sum_a N_a n^a .$$

Note that this integer number is purely topological and does not contain the elasticity tetrads.

The mixed anomaly corresponding to Eq. (10) in $D = 4$ can be related to a topological term in $D = 6$ dimensional spacetime in terms of the torsion in Eq. (13) and $U(1)$ gauge field:

$$S_{6D} = \frac{1}{96\pi^2} \sum_{a=1}^{n} N_a \int\limits_{X^6} d^6x \, \epsilon^{\mu\nu\alpha\beta\gamma\delta} T^a_{\gamma\delta} F_{\mu\nu} F_{\alpha\beta} \,. \quad (16)$$

This anomaly is mixed, since it contains the real or effective gauge field acting on Weyl fermions, and the tetrads of elasticity gravity.

In Eq. (16) the number n of the tetrad components ${E_\gamma}^a$ describing the crystal "planes" can be smaller than the dimension of space (if in the other, missing directions the system is not periodic). This happens for example for smectic liquid crystals, where there is one set of planes $n = 1$, and for vortex lattices, [27] with two sets of planes, $n = 2$. Here $n = 3$ for the 3+1d topological insulators, while for e. g. Wilson fermions [28] and in Floquet crystals [29, 30], one has $n = 4$.

Dimensional reduction of Eq. (16) to $D = 5$ gives the mixed Wess–Zumino term:

$$S_{5D} = \frac{1}{8\pi^2} \sum_{a=1}^{n} N_a \int_{X^5} d^5x\, \epsilon^{\mu\nu\alpha\beta\gamma} {E_\gamma}^a F_{\mu\nu} F_{\alpha\beta}\,. \quad (17)$$

In the absence of dislocations, the 5-form in the integrand of Eq. (17) is not only a closed but an exact form, and transforms to a surface integral, i. e. to Eq. (10). This describes the dimensional reduction of Eq. (16) to the $D = 4$ action in Eq. (10) via the $D = 5$ Wess–Zumino term in Eq. (17).

4. CONCLUSIONS

The weak topological quantum Hall insulators in 3+1-dimensions are described by the anomalous quantum Hall effect and Chern–Simons term in the action in Eq. (10). The AQHE response contains three elasticity vielbein fields ${E_\mu}^a(x)$, with $a = 1, 2, 3$. In the absence of dislocations, the Chern–Simons action is gauge invariant. In the presence of dislocations, the anomaly cancellation is produced by the Callan–Harvey mechanism [22–25].

The vielbeins ${E_\mu}^a(x)$ entering the response have the same dimension as the vector-potential of the gauge field, i.e. the dimension $[L]^{-1}$. This is consistent with the fact, that in the presence of dislocations, the three vielbein fields become the torsion gauge fields of the local $U(1) \times U(1) \times U(1)$ group of broken translation symmetry of crystals modulo the periodicity of the Bravais lattice [21]. The field strengths of these gauge fields correspond to the torsion, expressed via tetrads according to Eq. (13). In general relativity, such description of vielbein in terms of the Lie group can be found in Ref. [3]. Gravity based on tetrad fields with dimension $[L]^{-1}$ may occur also in the case that the microscopic structure of our quantum vacuum is periodic. However, at the moment it is clear that the Lorentz invariance persists at energy scale larger than the Planck scale [31].

In connection between the anomalies in spacetime dimensions $D = 2n + 2$, $D = 2n + 1$ and $D = 2n$, the weak topological insulator system under consideration corresponds to $n = 2$. In the presence of dislocations, the system can be described by the mixed 5D Wess–Zumino term in Eq. (17), which in turn can be obtained by dimensional reduction from the 6D mixed anomaly term in Eq. (16). The response is sensitive to the topology in momentum-frequency space and the prefactor of the Chern-Simons and Wess–Zumino actions contains three integer topological invariants. These are the winding number 3-forms expressed in terms of the Green's function in momentum-frequency space. We also noted that for Floquet systems, a 4D periodic description with four invariants becomes possible.

Concerning the Weyl semimetals and Weyl superfluids and superconductors, we pointed out that for a general $q_\mu(x)$, the action in Eq. (12) is not gauge invariant. In the presence of deformations of the parameter $q_\mu(x)$, an action corresponding to Eq. (12) would violate gauge invariance. It is gauge invariant only if q_μ is a constant of Nature in the corresponding topological vacuum. In this case, the Fermi arcs corresponding to the Weyl nodes can be seen by anomaly inflow in the presence of a domain wall in $q_\mu(x)$ [29]. However, the Fermi arcs have a more robust derivation under deformations of the Fermi-surface and in terms of momentum conserving boundary conditions for the Weyl fermions in the semimetal [32, 33]. This is in contrast to the chiral anomaly of Weyl fermions that has been experimentally observed in superfluid ^{3}He-A [34]. Distinct from the 3+1d weak topological insulators with topological AQHE, the chiral anomaly of Weyl fermions cannot be represented in terms of a mixed 4D Chern–Simons action but arises from a 5D Wess–Zumino term of the effective gauge field of the orbital degrees of freedom with a integer quantization of the coefficient of the resultant topological term [35].

In the case considered here, the action Eq. (10) contains the elasticity tetrads, which makes the action gauge invariant. The same action, if it is expressed in terms of the Weyl fermion tetrads, looses the gauge invariance. This demonstrates the principal difference between the elasticity tetrads and the Weyl fermion tetrads.

The effective gravity in terms of the Weyl tetrads emerging near the left handed and right handed chiral fermions in condensed matter prompts the following question for the interplay of chiral fermions of Standard Model and gravity: It is possible that both gravity and the Weyl fermions of Standard Model are the fundamental phenomena. However, it is not excluded that both of them are emergent, or one of them is more fundamental than the other.

In the latter case there are two scenarios, depending on what is more fundamental: gravity or the chiral Weyl fermions. If gravity is more fundamental, the tetrads should be the same for left and right fermions. This means that the Pauli matrices σ^a with $a = 1, 2, 3$ must have different signs for right and left handed fermions. Conversely, if the chiral fermions are the primary objects, then the tetrads should be different for left and right handed fermions. But they must be connected by a (discrete) symmetry, in order to obtain the PT-invariant Dirac equation where the two components mix, and to produce the same metric field (standard unimetric gravity).

Finally, here we did not consider the interplay of the Weyl tetrad gravity and the elasticity tetrad gravity [11, 36–38], which in principle may lead to new topological invariants, expressed for example via the combination of the two torsion fields.

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